Diffusion of a massive quantum particle coupled to a quasi-free thermal medium.

W. De Roeck[1]

Institute for Theoretical Physics
K.U.Leuven
B3001 Heverlee, Belgium

Institute for Theoretical Physics
ETH Zürich
CH-8093 Zürich, Switzerland

J. Fröhlich

Institute for Theoretical Physics
ETH Zürich
CH-8093 Zürich, Switzerland

Abstract: We consider a heavy quantum particle with an internal degree of freedom moving on the d-dimensional lattice $\mathbb{Z}^d$ (e.g., a heavy atom with finitely many internal states). The particle is coupled to a thermal medium (bath) consisting of free relativistic bosons (photons or Goldstone modes) through an interaction of strength λ linear in creation and annihilation operators. The mass of the quantum particle is assumed to be of order λ^{-2}, and we assume that the internal degree of freedom is coupled "effectively" to the thermal medium. We prove that the motion of the quantum particle is diffusive in $d \geq 4$ and for λ small enough.

KEY WORDS: diffusion, weak coupling limit, quantum Boltzmann equation, quantum field theory

1 Introduction

1.1 Diffusion

Diffusion and Brownian motion are central phenomena in the theory of transport processes and nonequilibrium statistical physics in general. One can think of the diffusion of a tracer particle in interacting particle systems, the diffusion of energy in coupled oscillator chains, and many other examples.

From a heuristic point of view, diffusion is rather well-understood in most of these examples. It can often be successfully described by some Markovian approximation, e.g. the Boltzmann equation or Fokker-Planck equation, depending on the example under study. In fact, this has been the strategy of Einstein in his ground breaking work of 1905, in which he modeled diffusion as a random walk.

However, up to this date, there is no rigorous derivation of diffusion from classical Hamiltonian mechanics or unitary quantum mechanics, except for some special chaotic systems; see Section 1.3.1. Such a derivation ought to allow us, for example, to prove that the motion of a tracer particle that interacts with its environment is diffusive at large times. In other words, one would like to prove a central limit theorem for the position of such a particle.

In recent years, some promising steps towards this goal have been taken. We provide a brief review of previous results in Section 1.3. In the present paper, we rigorously exhibit diffusion for a quantum particle weakly coupled to a thermal reservoir. However, our method is restricted to spatial dimension $d \geq 4$.

[1]Postdoctoral Fellow FWO-Flanders at K.U.Leuven, Belgium, email: `wojciech.deroeck@fys.kuleuven.be`

1.2 Informal description of the model and main results

We consider a quantum particle hopping on the lattice $\mathbb{Z}^d$, and interacting with a reservoir of bosons (photons or phonons) at temperature $\beta^{-1} > 0$. In the present section, we describe the system in a way that is appropriate at zero temperature, but is formal when $\beta < \infty$. The total Hilbert space, $\mathscr{H}$, of the coupled system is a tensor product of the particle space, $\mathscr{H}_{\mathrm{S}}$, with a reservoir space, $\mathscr{H}_{\mathrm{R}}$. Thus

$$\mathscr{H} := \mathscr{H}_{\mathrm{S}} \otimes \mathscr{H}_{\mathrm{R}}. \tag{1.1}$$

The particle space $\mathscr{H}_{\mathrm{S}}$ is given by $l^2(\mathbb{Z}^d) \otimes \mathscr{S}$, where the Hilbert space $\mathscr{S}$ describes the internal degrees of freedom of the particle, e.g., a (pseudo-)spin or dipole moment, and the particle Hamiltonian is given by the sum of the kinetic energy and the energy of the internal degrees of freedom

$$H_{\mathrm{S}} := H_{\mathrm{S,kin}} \otimes 1 + 1 \otimes H_{\mathrm{S,spin}} \tag{1.2}$$

The kinetic energy is chosen to be small in comparison with the interaction energy, and this is made manifest in its definition by a factor λ^2, where λ is the coupling strength between the particle and the reservoir (to be introduced below). Hence we set

$$H_{\mathrm{S,kin}} = \lambda^2 \varepsilon(P), \tag{1.3}$$

where the function ε is the dispersion law of the particle and P is the lattice-momentum operator. The most natural choice is to take $\varepsilon(P)$ to be (minus) the discrete lattice Laplacian, $-\Delta$. The energy of states of the internal degree of freedom is to a large extent arbitrary

$$H_{\mathrm{S,spin}} := Y, \qquad \text{for some Hermitian matrix } Y, \tag{1.4}$$

the main requirement being that Y not be equal to a multiple of the identity.

The reservoir is described by a free boson field; creation and annihilation operators creating/annihilating bosons with momentum $q \in \mathbb{R}^d$ are written as a_q^*, a_q, respectively. They satisfy the canonical commutation relations

$$[a_q^{\#}, a_{q'}^{\#}] = 0, \qquad [a_q, a_{q'}^*] = \delta(q - q'), \tag{1.5}$$

where $a^{\#}$ stands for either a or a^*. The energy of a reservoir mode q is given by the dispersion law $\omega(q) \geq 0$. To describe the coupling of the particle to the reservoir, we introduce a Hermitian matrix W on $\mathscr{S}$ and we write X for the position operator on $l^2(\mathbb{Z}^d)$.

The total Hamiltonian of the system is taken to be

$$H_\lambda := H_{\mathrm{S}} + \int_{\mathbb{R}^d} \mathrm{d}q\, \omega(q) a_q^* a_q + \lambda \int \mathrm{d}q \left(\mathrm{e}^{\mathrm{i}q\cdot X} \otimes W \otimes \phi(q) a_q + \mathrm{e}^{-\mathrm{i}q\cdot X} \otimes W \otimes \overline{\phi(q)} a_q^* \right) \tag{1.6}$$

acting on $\mathscr{H}_{\mathrm{S}} \otimes \mathscr{H}_{\mathrm{R}}$. The function $\phi(q)$ is a form factor and $\lambda \in \mathbb{R}$ is the coupling strength. We write H_{S} instead of $H_{\mathrm{S}} \otimes 1$, etc.

We introduce three important assumptions:

1) The kinetic energy is small w.r.t. the coupling term in the Hamiltonian, as has already been indicated by the inclusion of λ^2 in the definition of $H_{\mathrm{S,kin}}$. Physically, this means that the particle is heavy.
2) We require a linear dispersion law for the reservoir modes, $\omega(q) \equiv |q|$, in order to have good decay estimates at low speed. This means that the reservoir consists of photons, phonons or Goldstone modes of a Bose-Einstein condensate.
3) We assume that the amplitude of the wave front of a reservoir excitation (located on the light cone) has integrable (in time) decay. This is satisfied if the dimension of space[2] is at least 4.

[2] Since the integrability in time is only needed for reservoir excitations, we can in principle also treat models in which the particle is 3-dimensional, but the reservoir is effectively 4-dimensional.

Additional assumptions will concern the smoothness of the form factor ϕ and the "effectiveness" of the coupling to the heat bath (e.g., the interaction between the internal degrees of freedom and the reservoir, described by the matrix W, should not vanish.)

The initial state, $\rho^{\beta}_{\mathrm{R}}$, of the reservoir is chosen to be an equilibrium state at temperature $\beta^{-1} > 0$. For mathematical details on the construction of infinite reservoirs, see e.g. [10, 4, 2]. The initial state of the whole system, consisting of the particle and the reservoir, is a product state $\rho_{\mathrm{S}} \otimes \rho^{\beta}_{\mathrm{R}}$, with ρ_{S} a density matrix for the particle that will be specified later. The time-evolved density matrix of the particle ('subsystem') is called $\rho_{\mathrm{S},t}$ and is obtained by "tracing out the reservoir degrees of freedom" after the time-evolution has acted on the initial state during a time t, i.e., formally,

$$\rho_{\mathrm{S},t} := \mathrm{Tr}_{\mathscr{H}_{\mathrm{R}}} \left[\mathrm{e}^{-itH_{\lambda}} \left(\rho_{\mathrm{S}} \otimes \rho^{\beta}_{\mathrm{R}} \right) \mathrm{e}^{itH_{\lambda}} \right], \tag{1.7}$$

where $\mathrm{Tr}_{\mathscr{H}_{\mathrm{R}}}$ is the partial trace over $\mathscr{H}_{\mathrm{R}}$. We warn the reader that the above formula does not make sense mathematically for an infinitely extended reservoir, since the reservoir state $\rho^{\beta}_{\mathrm{R}}$ is not a density matrix on $\mathscr{H}_{\mathrm{R}}$. This is a consequence of the fact that the reservoir is described from the start in the thermodynamic limit and, hence, the reservoir modes form a continuum. Nevertheless, the LHS of formula (1.7) can be given a meaning in the thermodynamic limit.

The density matrix $\rho_{\mathrm{S},t}$ obviously depends on the coupling strength λ, but we do not indicate this explicitly. We also drop the subscript S and we simply write ρ_t, instead of $\rho_{\mathrm{S},t}$, in what follows.

We will often represent ρ_t as a $\mathscr{B}(\mathscr{S})$-valued function on $\mathbb{Z}^d \times \mathbb{Z}^d$:

$$\rho_t(x_L, x_R) \in \mathscr{B}(\mathscr{S}), \qquad x_L, x_R \in \mathbb{Z}^d. \tag{1.8}$$

Although this is not necessary for many of our results, we require the initial state of the particle to be exponentially localized near the origin of the lattice, i.e.,

$$\|\rho_t(x_L, x_R)\|_{\mathscr{B}(\mathscr{S})} \leq C \mathrm{e}^{-\delta'|x_L|} \mathrm{e}^{-\delta'|x_R|}, \qquad \text{for some constants } C, \delta' > 0 \tag{1.9}$$

Our first result concerns the diffusion of the position of the particle.

1.2.1 Diffusion

We define the probability density

$$\mu_t(x) := \mathrm{Tr}_{\mathscr{S}}\, \rho_t(x, x) \tag{1.10}$$

where $\mathrm{Tr}_{\mathscr{S}}$ denotes the partial trace over the internal degrees of freedom. The number $\mu_t(x)$ is the probability to find the particle at site x after time t.

By diffusion, we mean that, for large t,

$$\mu_t(x) \sim \left(\frac{1}{2\pi t} \right)^{d/2} (\det D)^{-1/2} \exp\{ -\frac{1}{2} \left(\frac{x}{\sqrt{t}} \cdot D^{-1} \frac{x}{\sqrt{t}} \right) \}, \tag{1.11}$$

where the diffusion tensor $D \equiv D_{\lambda}$ is a strictly positive matrix with real entries; actually, if the particle dispersion law ε is invariant under lattice rotations, then the tensor D is isotropic and hence a scalar. The magnitude of D is inferred from the following reasoning: The particle undergoes collisions with the reservoir modes. Let t_m be the mean time between two collisions, and let v_m be the mean speed of the particle (the direction of the particle velocity is assumed to be random). Then the mean free path is $v_m \times t_m$ and the central limit theorem suggests that the particle diffuses with diffusion constant

$$D \sim \frac{(v_m \times t_m)^2}{t_m} \tag{1.12}$$

The mean time t_m is of order $t_m \sim \lambda^{-2}$ since the interaction with the reservoir contributes only in second order. The mean velocity v_m is of order $v_m \sim \lambda^2$ because of the factor λ^2 in the definition of the kinetic energy. Hence $D \sim \lambda^2$.

We now move towards quantifying (1.11). Let us fix a time t. Since $\mu_t(x)$ is a probability measure, one can think of x_t as a random variable with

$$\mathrm{Prob}(x_t = x) := \mu_t(x). \tag{1.13}$$

The claim that the random variable $\frac{x_t}{\sqrt{t}}$ converges in distribution, as $t \nearrow \infty$, to a Gaussian random variable with mean 0 and variance D is called a Central Limit Theorem (CLT). It is equivalent to pointwise convergence of the characteristic function, i.e.,

$$\sum_{x\in\mathbb{Z}^d} \mathrm{e}^{-\frac{\mathrm{i}}{\sqrt{t}}x\cdot q}\mu_t(x) \quad \underset{t\uparrow\infty}{\longrightarrow} \quad \mathrm{e}^{-\frac{1}{2}q\cdot Dq}, \quad \text{for all } q \in \mathbb{R}^d, \tag{1.14}$$

and it is this statement which is our main result, Theorem 3.1. A stronger version of the convergence in (1.14) (also included in Theorem 3.1) implies that the rescaled moments of μ_t converge. For example, for $i, j = 1, \ldots, d$,

$$\frac{1}{t}\,\mathrm{Tr}[\rho_t X_i] = \frac{1}{t}\sum_x x_i \mu_t(x) \qquad \underset{t\uparrow\infty}{\longrightarrow} \qquad 0 \tag{1.15}$$

$$\frac{1}{t}\,\mathrm{Tr}[\rho_t X_i X_j] = \frac{1}{t}\sum_x x_i x_j \mu_t(x) \qquad \underset{t\uparrow\infty}{\longrightarrow} \qquad D_{i,j}, \tag{1.16}$$

In fact, the first line (vanishing of average drift) is expected only if one assumes that the model has space inversion symmetry, which is assumed throughout.

1.2.2 Equipartition

Our second result concerns the asymptotic expectation value of the kinetic energy of the particle and the internal degrees of freedom. The equipartition theorem suggests that the energy of all degrees of freedom of the particle, the translational and internal degrees of freedom, thermalizes at the temperature β^{-1} of the heat bath. We will establish this property up to a correction that is small in the coupling strength λ. This is acceptable, since the interaction effectively modifies the Gibbs state of the particle. We prove that, for all bounded functions F,

$$\rho_t(F(H_{\mathrm{S},kin})) \qquad \underset{t\nearrow\infty}{\longrightarrow} \quad \frac{1}{Z}\int_{\mathbb{T}^d} \mathrm{d}k\; F(\lambda^2\varepsilon(k))\mathrm{e}^{-\beta\lambda^2\varepsilon(k)} + o(|\lambda|^0) \tag{1.17}$$

$$\rho_t(F(H_{\mathrm{S},spin})) \qquad \underset{t\nearrow\infty}{\longrightarrow} \qquad \frac{1}{Z'}\sum_{e\in\mathrm{sp}Y} F(e)\mathrm{e}^{-\beta e} + o(|\lambda|^0), \qquad \text{as } \lambda \searrow 0 \tag{1.18}$$

where Z, Z' are normalization constants and the sum $\sum_{e\in\mathrm{sp}Y}$ ranges over all eigenvalues of the Hamiltonian Y. We note that the factor $\mathrm{e}^{-\beta\lambda^2\varepsilon(k)}$ can be replaced by 1 (as in Theorem 3.2) since we anyhow allow a correction term that is small in λ and the function $\varepsilon(k)$ is bounded. For this reason, one could say that, for very small values of λ, the translational degrees of freedom thermalize at infinite temperature ($\beta = 0$).

1.2.3 Decoherence

By decoherence we mean that off-diagonal elements $\rho_t(x,y)$ of the density matrix ρ_t in the position representation fall off rapidly in the distance between x and y. Of course, this property can only hold at large enough times when the effect of the reservoir on the particle has destroyed all initial long-distance coherence, i.e., after a time of order λ^{-2}. Thus, there is a decoherence length $1/\gamma_{dch}$ and a decay rate g such that

$$\|\rho_t(x_L, x_R)\|_{\mathscr{B}(\mathscr{S})} \leq C\mathrm{e}^{-\gamma_{dch}|x_L - x_R|} + C'\mathrm{e}^{-\lambda^2 gt}, \qquad \text{as } t \nearrow \infty \tag{1.19}$$

for some constants C, C'. The magnitude of the inverse decoherence rate γ_{dch} is determined as follows: The time the reservoir needs to destroy coherence is of the order of the mean free time t_m, while the time that is needed for coherence to be built up over a distance $1/\gamma_{dch}$ is given by $(\gamma_{dch} \times v_m)^{-1}$, where v_m is the mean velocity of the particle. Equating these two times yields

$$\gamma_{dch} \sim (t_m \times v_m)^{-1} \tag{1.20}$$

and hence, recalling that $t_m \sim \lambda^{-2}$ and $v_m \sim \lambda^2$, as argued in Section 1.2.1, we find that γ_{dch} does not scale with λ.

1.3 Related results and discussion

1.3.1 Classical mechanics

Diffusion has been established for the two-dimensional finite horizon billiard in [6]. In that setup, a point particle travels in a periodic, planar array of fixed hard-core scatterers. The *finite-horizon condition* refers to the fact that the particle cannot move further than a fixed distance without hitting an obstacle.

Knauf [27] replaced the hard-core scatterers by a planar lattice of attractive Coulombic potentials, i.e., the potential is $V(x) = -\sum_{j\in\mathbb{Z}^2} \frac{1}{|x-j|}$. In that case, the motion of the particle can be mapped to the free motion on a manifold with strictly negative curvature, and one can again prove diffusion.

Recently, a different approach was taken in [24]: the authors of [24] consider a $d = 3$ lattice of confined particles that interact locally with chaotic maps such that the energy of the particles is preserved but their momenta are randomized. Neighboring particles can exchange energy via collisions and one proves diffusive behaviour of the energy profile.

1.3.2 Quantum mechanics for extended systems

The earliest result for extended quantum systems that we are aware of, [30], treats a quantum particle interacting with a time-dependent random potential that has no memory (the time-correlation function is $\delta(t)$). Recently, this was generalized in [25] to the case of time-dependent random potentials where the time-dependence is given by a Markov process with a gap (hence, the free time-correlation function of the environment is exponentially decaying). In [32], we treated a quantum particle interacting with independent heat reservoirs at each lattice site. This model also has an exponentially decaying free reservoir time-correlation function and as such, it is very similar to [25]. Notice also that, in spirit, the model with independent heat baths is comparable to the model of [24], but, in practice, it is easier since quantum mechanics is linear!

The most serious shortcoming of these results is the fact that the assumption of exponential decay of the correlation function in time is unrealistic. In the model of the present paper, the space-time correlation function, called $\psi(x,t)$ in what follows, is the correlation function of freely-evolving excitations in the reservoir, created by interaction with the particle. Since momentum is conserved locally, these excitations cannot decay exponentially in time t, uniformly in x. For example, if the dispersion law of the reservoir modes is linear, then $\psi(x,t)$ is a solution of the linear wave equation. In $d = 3$, it behaves qualitatively as

$$\psi(x,t) \sim \frac{1}{|x|}\delta(c|t| - |x|), \qquad \text{with } c \text{ the propagation speed of the reservoir modes} \tag{1.21}$$

In higher dimensions, one has better dispersive estimates, namely $\sup_x |\psi(x,t)| \leq t^{\frac{d-1}{2}}$ (under certain conditions), and this is the reason why, for the time being, our approach is restricted to $d \geq 4$. In the Anderson model, the analogue of the correlation function does not decay at all, since the potentials are fixed in time. Indeed, the Anderson model is different from our particle-reservoir model: diffusion is only expected to occur for small values of the coupling strength, whereas the particle gets trapped (Anderson localization) at large coupling.

Finally, we mention a recent and exciting development: in [13], the existence of a delocalized phase in three dimensions is proven for a supersymmetric model which is interpreted as a toy version of the Anderson model.

1.3.3 Quantum mechanics for confined systems

The theory of confined quantum systems, i.e., multi-level atoms, in contact with quasi-free thermal reservoirs has been intensively studied in the last decade, e.g. by [3, 23, 12]. In this setup, one proves approach to equilibrium for the multi-level atom. Although at first sight, this problem is different from ours (there is no analogue of diffusion), the techniques are quite similar and we were mainly inspired by these results. However, an important difference is that, due to its confinement, the multi-level atom experiences a free reservoir correlation function with better decay properties than that of our model. For example, in [23], the free reservoir correlation function is actually exponentially decaying.

1.3.4 Scaling limits

Up to now, most of the rigorous results on diffusion starting from deterministic dynamics are formulated in a *scaling limit*. This means that one does not fix one dynamical system and study its behaviour in the long-time limit, but, rather, one compares a family of dynamical systems at different times. The precise definition of the scaling limit differs from model to model, but, in general, one scales time, space and the coupling strength (and possibly also the initial state) such that the Markovian approximation to the dynamics becomes exact. In our model the natural scaling limit is the so-called weak coupling limit: one introduces the macroscopic time $\tau := \lambda^2 t$ and one takes the limit $\lambda \searrow 0, t \nearrow \infty$ while keeping τ fixed. In that limit, the dynamics of the particle becomes Markovian in τ (as if the heat bath had no memory) and it is described by a Lindblad evolution. The long-time behavior of this Lindblad evolution is diffusive. This is explained in detail in Section 4. One may say that, in this scaling limit, the heuristic reasoning employed in the previous sections to deduce the λ-dependence of the diffusion constant and the decoherence length becomes exact. The same scaling is known very well in the theory of confined open quantum systems as it gives rise to the Pauli master equation. This was first made precise in [9].

If we had set up the model with a kinetic energy of $O(1)$ (instead of $O(\lambda^2)$), then one should also rescale space by introducing the macroscopic space-coordinate $\chi := \lambda^2 x$. The reason for this additional rescaling is that, between two collisions, a particle with mass of order 1 moves during a time of order λ^{-2}, and hence it travels a distance of order λ^{-2}. The resulting scaling limit

$$x \to \lambda^{-2} x, \quad t \to \lambda^{-2} t, \quad \lambda \searrow 0 \tag{1.22}$$

is often called the *kinetic limit*. In the kinetic limit the dynamics of the particle is described by a linear Boltzmann equation (LBE) in the variables (χ, τ). The convergence of the particle dynamics to the LBE has been proven in [14] for a quantum particle coupled to a heat bath, and in [17] for a quantum particle coupled to a random potential (Anderson model). The long-time, large-distance limit of the Boltzmann equation is the heat equation, which suggests that one should be able to derive the heat equation directly in the limiting regime corresponding to

$$x \to \lambda^{-(2+\kappa)} x, \quad t \to \lambda^{-(2+2\kappa)} t, \quad \lambda \searrow 0, \qquad \text{for some } \kappa > 0. \tag{1.23}$$

This was accomplished in [16, 15] for the Anderson model. An analogous result was obtained in [28] for a classical particle moving in a random force field.

1.3.5 Limitations to our result

Two striking features of our model are the large mass, of order λ^{-2}, and the internal degrees of freedom described by the Hamiltonian $H_{\mathrm{S,spin}} = Y$. Physically speaking, these choices are of course not necessary for diffusion, they just make our task of proving it easier. Let us explain why this is so. First of all, once the mass is chosen to be of order λ^{-2}, the internal degrees of freedom are necessary to make the model diffusive in second order perturbation theory. Without the internal degrees of freedom, it would be ballistic. This is explained in Section 4.2; in particular, it can be deduced immediately from conservation of momentum and energy for the processes in Figure 3. Note also that the dependence on λ is chosen such that the kinetic term $H_{\mathrm{S,kin}} = \lambda^2 \varepsilon(P)$ is comparable to the particle-reservoir interaction in second order of perturbation theory (both are of order λ^2). The large mass ensures that the position of the particle remains well-defined for a time of order λ^{-2}, which permits us to sum up Feynman diagrams in real space.

Further, we note that our result requires an analyticity assumption on the form factor ϕ, see Assumption 2.3. This assumption ensures that the free reservoir correlation function $\psi(x, t)$ is exponentially decaying for small x, even though it has slow decay on the lightcone, as explained in Section 1.3.2.

1.4 Outline of the paper

The model is introduced in Section 2 and the results are stated in Section 3. In Section 4, we describe the Markovian approximation to our model. This approximation provides most of the intuition and it is a key ingredient of the proofs. Section 5 describes the main ideas of the proof, which is contained in the remaining Sections 6-9 and the four appendices A-D.

Acknowledgements

W. De Roeck thanks J. Bricmont and H. Spohn for helpful discussions and suggestions, and for pointing out several references. He has also greatly benefited from collaboration with J. Clark and C. Maes. In particular, the results described in Section 4 and Appendix C were essentially obtained in [7]. After the first version was submitted, some inaccuracies were pointed out by L. Erdös, A. Knowles, H.-T. Yau and J. Yin. Most importantly, the remarks of an anonymous referee allowed for serious improvements in the presentation of the proof. Finally, W.D.R. acknowledges the financial support of the Flemish research fund 'FWO-Vlaanderen'.

2 The model

After fixing conventions in Section 2.1, we introduce the model. Section 2.2 describes the particle, while Section 2.3 deals with the reservoir. In Section 2.4, we couple the particle to the reservoir, and we define the reduced particle dynamics $\mathcal{Z}_t$. Section 2.5 introduces the fiber decomposition.

2.1 Conventions and notation

Given a Hilbert space $\mathscr{E}$, we use the standard notation

$$\mathscr{B}_p(\mathscr{E}) := \left\{ S \in \mathscr{B}(\mathscr{E}), \mathrm{Tr}\left[(S^*S)^{p/2}\right] < \infty \right\}, \qquad 1 \leq p \leq \infty, \tag{2.1}$$

with $\mathscr{B}_\infty(\mathscr{E}) \equiv \mathscr{B}(\mathscr{E})$ the bounded operators on $\mathscr{E}$, and

$$\|S\|_p := \left(\mathrm{Tr}\left[(S^*S)^{p/2}\right]\right)^{1/p}, \qquad \|S\| := \|S\|_\infty. \tag{2.2}$$

For bounded operators acting on $\mathscr{B}_p(\mathscr{E})$, i.e. elements of $\mathscr{B}(\mathscr{B}_p(\mathscr{E}))$, we use in general the calligraphic font: $\mathcal{V}, \mathcal{W}, \mathcal{T}, \ldots$. An operator $X \in \mathscr{B}(\mathscr{E})$ determines an operator $\mathrm{ad}(X) \in \mathscr{B}(\mathscr{B}_p(\mathscr{E}))$ by

$$\mathrm{ad}(X)S := [X, S] = XS - SX, \qquad S \in \mathscr{B}_p(\mathscr{E}). \tag{2.3}$$

The norm of operators in $\mathscr{B}(\mathscr{B}_p(\mathscr{E}))$ is defined by

$$\|\mathcal{W}\| := \sup_{S \in \mathscr{B}_p(\mathscr{E})} \frac{\|\mathcal{W}(S)\|_p}{\|S\|_p}. \tag{2.4}$$

We will mainly work with Hilbert-Schmidt operators ($p = 2$) and, unless mentioned otherwise, the notation $\|\mathcal{W}\|$ will refer to this case.

For vectors $v \in \mathbb{C}^d$, we let $\mathrm{Re}\, v, \mathrm{Im}\, v$ denote the vectors $(\mathrm{Re}\, v_1, \ldots, \mathrm{Re}\, v_d)$ and $(\mathrm{Im}\, v_1, \ldots, \mathrm{Im}\ v_d)$, respectively. The scalar product on $\mathbb{C}^d$ is written as $v \cdot v'$ and the norm as $|v| := \sqrt{v \cdot v}$.

The scalar product on a general Hilbert space $\mathscr{E}$ is written as $\langle \cdot, \cdot \rangle$, or, occasionally, as $\langle \cdot, \cdot \rangle_{\mathscr{E}}$. All scalar products are defined to be linear in the second argument and anti-linear in the first one. We use the physicists' notation

$$|\varphi\rangle\langle\varphi'| \qquad \text{for the rank-1 operator in } \mathscr{B}(\mathscr{E}) \text{ acting as} \quad \varphi'' \mapsto \langle \varphi', \varphi'' \rangle \varphi \tag{2.5}$$

We write $\Gamma_{\mathrm{s}}(\mathscr{E})$ for the symmetric (bosonic) Fock space over the Hilbert space $\mathscr{E}$ and we refer to [10] for definitions and discussion. If ω is a self-adjoint operator on $\mathscr{E}$, then its (self-adjoint) second quantization, $\mathrm{d}\Gamma_{\mathrm{s}}(\omega)$, is defined by

$$\mathrm{d}\Gamma_{\mathrm{s}}(\omega)\mathrm{Sym}(\varphi_1 \otimes \ldots \otimes \varphi_n) := \sum_{i=1}^{n} \mathrm{Sym}(\varphi_1 \otimes \ldots \otimes \omega\varphi_i \otimes \ldots \otimes \varphi_n), \tag{2.6}$$

where Sym projects on the symmetric subspace of $\otimes^n \mathscr{E}$ and $\varphi_1, \ldots, \varphi_n \in \mathscr{E}$.

We use C, C' to denote constants whose precise value can change from equation to equation.

2.2 The particle

We choose a finite-dimensional Hilbert space $\mathscr{S}$, which can be thought of as the state space of some internal degrees of freedom of the particle, such as spin or a dipole moment. The total Hilbert space of the particle is given by $\mathscr{H}_{\mathrm{S}} := l^2(\mathbb{Z}^d, \mathscr{S}) = l^2(\mathbb{Z}^d) \otimes \mathscr{S}$ (the subscript S refers to 'system', as is customary in system-reservoir models).

We define the position operators, X_j, on $\mathscr{H}_{\mathrm{S}}$ by

$$(X_j\varphi)(x) = x_j\varphi(x), \qquad x \in \mathbb{Z}^d, \quad \varphi \in l^2(\mathbb{Z}^d, \mathscr{S}), \quad j = 1, \dots, d \tag{2.7}$$

In what follows, we will almost always drop the component index j and write $X \equiv (X_j)$ to denote the vector-valued position operator. We will often consider the space $\mathscr{H}_{\mathrm{S}}$ in its dual representation, i.e. as $L^2(\mathbb{T}^d, \mathscr{S})$, where $\mathbb{T}^d$ is the d-dimensional torus (momentum space), which is identified with $L^2([-\pi, \pi]^d, \mathscr{S})$. We formally define the 'momentum' operator P as multiplication by $k \in \mathbb{T}^d$, i.e.,

$$P\varphi(k) = k\varphi(k), \quad k \in [-\pi, \pi]^d, \varphi \in L^2(\mathbb{T}^d, \mathscr{S}) \tag{2.8}$$

Although P is well-defined as a bounded operator, it does not correspond to a continuous function on $\mathbb{T}^d$, and it is not true that $[X_j, P_j] = -\mathrm{i}$. Throughout the paper, we will only use operators $F(P)$ where F is a function on $\mathbb{T}^d$ that is extended periodically to $\mathbb{R}^d$. We choose such a periodic function, ε, of P to determine the dispersion law of the particle. The kinetic energy of our particle is given by $\lambda^2\varepsilon(P)$, where λ is a small parameter, i.e., the 'mass' of the particle is of order λ^{-2}

The energy of the internal degrees of freedom is given by a self-adjoint operator $Y \in \mathcal{B}(\mathscr{S})$, acting on $\mathscr{H}_{\mathrm{S}}$ as $(Y\varphi)(k) = Y(\varphi(k))$. The Hamiltonian of the particle is

$$H_{\mathrm{S}} := \lambda^2\varepsilon(P) \otimes 1 + 1 \otimes Y \tag{2.9}$$

As in Section 1, we will mostly write $\varepsilon(P)$ instead of $\varepsilon(P) \otimes 1$ and Y instead of $1 \otimes Y$.

Our first assumption ensures that the Hamiltonian $H_{\mathrm{S}} = Y + \lambda^2\varepsilon(P)$ has good regularity properties

Assumption 2.1 (Analyticity of the particle dynamics)**.** *The function ε, defined originally on $\mathbb{T}^d$, extends to an analytic function in a neighborhood of the complex multistrip of width $\delta_\varepsilon > 0$. That is, when viewed as a periodic function on $\mathbb{R}^d$, ε is analytic (and bounded) in a neighborhood of $(\mathbb{R} + \mathrm{i}[-\delta_\varepsilon, \delta_\varepsilon])^d$. Moreover, ε is symmetric with respect to space inversion, i.e.,*

$$\varepsilon(k) = \varepsilon(-k). \tag{2.10}$$

Furthermore, we assume there is no $v \in \mathbb{R}^d$ such that the function $k \mapsto v \cdot \nabla\varepsilon(k)$ vanishes identically and that ε does not have a smaller periodicity than that of $\mathbb{T}^d$, i.e., we assume that

$$\varepsilon(k) = \varepsilon(z + k) \textit{ for all } k \in \mathbb{T}^d \qquad \Leftrightarrow \qquad z \in (2\pi\mathbb{Z})^d. \tag{2.11}$$

The most natural choice for ε is $\varepsilon(k) = \sum_{i=1}^d 2(1 - \cos(k_i))$, which corresponds to $-\varepsilon(P)$ being the lattice Laplacian. As already indicated in Section 1.2.1, the symmetry assumption (2.10) is necessary to exclude an asymptotic drift of the particle.

By a simple Paley-Wiener argument, Assumption 2.1 implies that one has exponential propagation estimates for the evolution generated by the operator $\varepsilon(P)$. Indeed, from the relation

$$\left\|(\mathrm{e}^{\mathrm{i}\nu\cdot X}\mathrm{e}^{-\mathrm{i}t\varepsilon(P)}\mathrm{e}^{-\mathrm{i}\nu\cdot X})\right\| = \left\|\mathrm{e}^{-\mathrm{i}t\varepsilon(P+\nu)}\right\| \leq \mathrm{e}^{q_\varepsilon(|\mathrm{Im}\,\nu|)|t|}, \qquad \text{for } |\mathrm{Im}\,\nu| < \delta_\varepsilon \tag{2.12}$$

with $q_\varepsilon(\gamma) := \sup_{|\mathrm{Im}\,p| \leq \gamma} |\mathrm{Im}\,\varepsilon(p)|$, one obtains

$$\left\|(\mathrm{e}^{-\mathrm{i}t\varepsilon(P)})(x_L, x_R)\right\|_{\mathscr{S}} \leq \mathrm{e}^{-\gamma|x_L - x_R|}\mathrm{e}^{q_\varepsilon(\gamma)|t|}, \qquad \text{for } \gamma \leq \delta_\varepsilon \tag{2.13}$$

where we write $S(x_L, x_R)$ for a $\mathscr{B}(\mathscr{S})$-valued 'matrix element' of $S \in \mathscr{B}(\mathscr{H}_{\mathrm{S}})$.

2.3 The reservoirs

2.3.1 The reservoir space

We introduce a one-particle reservoir space $\mathfrak{h} = L^2(\mathbb{R}^d)$ and a positive one-particle Hamiltonian $\omega \geq 0$. The coordinate $q \in \mathbb{R}^d$ should be thought of as a momentum coordinate, and ω acts by multiplication with a function $\omega(q)$,

$$(\omega\varphi)(q) = \omega(q)\varphi(q) \tag{2.14}$$

In other words, ω is the dispersion law of the reservoir particles. The full reservoir Hilbert space, $\mathscr{H}_{\mathrm{R}}$, is the symmetric Fock space (see Section 2.1 or [10]) over the one-particle space $\mathfrak{h}$,

$$\mathscr{H}_{\mathrm{R}} := \Gamma_{\mathrm{s}}(\mathfrak{h}) \tag{2.15}$$

The reservoir Hamiltonian, H_{R}, acting on $\mathscr{H}_{\mathrm{R}}$, is then the second quanitzation of ω

$$H_{\mathrm{R}} := \mathrm{d}\Gamma_{\mathrm{s}}(\omega) = \int_{\mathbb{R}^d} \mathrm{d}q\, \omega(q) a_q^* a_q. \tag{2.16}$$

with the creation/annihilation operators a_q^*, a_q to be introduced below.

2.3.2 The system-reservoir coupling

The coupling between system and reservoir is assumed to be translation invariant. We choose a 'form factor' $\phi \in L^2(\mathbb{R}^d)$ and a self-adjoint operator $W = W^* \in \mathscr{B}(\mathscr{S})$ with $\|W\| \leq 1$, and we define the interaction Hamiltonian H_{SR} by

$$H_{\mathrm{SR}} := \int \mathrm{d}q \left(\mathrm{e}^{\mathrm{i}q\cdot X} \otimes W \otimes \phi(q) a_q + \mathrm{e}^{-\mathrm{i}q\cdot X} \otimes W \otimes \overline{\phi(q)} a_q^* \right) \quad \text{on} \quad \mathscr{H}_{\mathrm{S}} \otimes \mathscr{H}_{\mathrm{R}}, \tag{2.17}$$

where a_q, a_q^* are the creation/annihilation operators on $\mathfrak{h}$ satisfying the canonical commutation relations (CCR)

$$[a_q, a_{q'}^*] = \delta(q - q'), \qquad [a_q^{\#}, a_{q'}^{\#}] = 0 \tag{2.18}$$

with $a^{\#}$ standing for either a or a^*. We also introduce the smeared creation/annihilation operators

$$a^*(\varphi) := \int_{\mathbb{R}^d} \mathrm{d}q\, \varphi(q) a_q^*, \qquad a(\varphi) := \int_{\mathbb{R}^d} \mathrm{d}q\, \overline{\varphi(q)} a_q, \qquad \varphi \in L^2(\mathbb{R}^d). \tag{2.19}$$

In what follows we will specify our assumptions on H_{SR}, but we already mention that we need $[W, Y] \neq 0$ for the internal degrees of freedom to be coupled effectively to the field.

2.3.3 Thermal states

Next, we put some tools in place to describe the positive temperature state of the reservoir. We introduce the density operator

$$T_\beta = (\mathrm{e}^{\beta\omega} - 1)^{-1} \quad \text{on} \quad \mathfrak{h} = L^2(\mathbb{R}^d). \tag{2.20}$$

Let $\mathfrak{C}$ be the *algebra consisting of polynomials in the creation and annihilation operators $a(\varphi), a^*(\varphi')$ with $\varphi, \varphi' \in \mathfrak{h}$. We define $\rho_{\mathrm{R}}^{\beta}$ as a quasi-free state defined on $\mathfrak{C}$. It is fully specified by the following properties:

1) Gauge-invariance

$$\rho_{\mathrm{R}}^{\beta}[a^*(\varphi)] = \rho_{\mathrm{R}}^{\beta}[a(\varphi)] = 0 \tag{2.21}$$

2) The choice of the two-particle correlation function

$$\begin{pmatrix} \rho_{\mathrm{R}}^{\beta}[a^*(\varphi)a(\varphi')] & \rho_{\mathrm{R}}^{\beta}[a^*(\varphi)a^*(\varphi')] \\ \rho_{\mathrm{R}}^{\beta}[a(\varphi)a(\varphi')] & \rho_{\mathrm{R}}^{\beta}[a(\varphi)a^*(\varphi')] \end{pmatrix} = \begin{pmatrix} \langle \varphi' | T_\beta \varphi \rangle & 0 \\ 0 & \langle \varphi | (1 + T_\beta) \varphi' \rangle \end{pmatrix} \tag{2.22}$$

3) The state $\rho_{\mathrm{R}}^{\beta}$ is quasifree. This means that the higher correlation functions are related to the two-particle correlation function via Wick's theorem

$$\rho_{\mathrm{R}}^{\beta}\left[a^{\#}(\varphi_1)\dots a^{\#}(\varphi_{2n})\right] = \sum_{\pi\in\mathcal{P}_n}\prod_{(i,j)\in\pi}\rho_{\mathrm{R}}^{\beta}\left[a^{\#}(\varphi_i)a^{\#}(\varphi_j)\right] \tag{2.23}$$

where $a^{\#}$ stands for either a^* or a, and $\mathcal{P}_n$ is the set of pairings π, partitions of $\{1,\dots,2n\}$ into n pairs (r,s). By convention, we fix the order within the pairs such that $r<s$.

The reason that it suffices to specify the state on $\mathfrak{C}$ has been explained in many places, see e.g. [4, 20, 10]

2.3.4 Assumptions on the reservoir

Next, we state our main assumption restricting the type of reservoir and the dimensionality of space.

Assumption 2.2 (Relativistic reservoir and $d\geq 4$)**.** *We assume that the*

$$\textit{dimension of space } d\geq 4 \tag{2.24}$$

Further, we assume the dispersion law of the reservoir particles to be linear;

$$\omega(q) := |q| \tag{2.25}$$

For simplicity, we will assume that the form factor ϕ is rotationally symmetric and we write

$$\phi(q)\equiv\phi(|q|), \qquad q\in\mathbb{R}^d \tag{2.26}$$

We define the "effective squared form factor" as

$$\hat{\psi}(\omega) := |\omega|^{(d-1)}\begin{cases}\frac{1}{1-\mathrm{e}^{-\beta\omega}}\,|\phi(|\omega|)|^2 & \omega\geq 0\\ \frac{1}{\mathrm{e}^{-\beta\omega}-1}\,|\phi(|\omega|)|^2 & \omega<0\end{cases} \tag{2.27}$$

where we are abusing the notation by letting ω denote a variable in $\mathbb{R}$. Previously, ω was the energy operator on the one-particle Hilbert space and as such, it could assume only positive values. Indeed, at positive temperature, the function $\hat{\psi}(\omega)$ plays a similar role as $|\phi(|\omega|)|^2$ at zero-temperature: It describes the intensity of the coupling to the reservoir modes of frequency ω. Modes with $\omega<0$ appear only at positive temperature and they correspond physically to "holes". One checks that $\frac{\hat{\psi}(\omega)}{\hat{\psi}(-\omega)}=\mathrm{e}^{\beta\omega}$, which is Einstein's emission-absorption law (i.e. detailed balance). This particle-hole point of view can be incorporated into the formalism by the Araki-Woods representation, see e.g. [4, 20, 10].

The next assumption restricts the "effective squared form factor" $\hat{\psi}$.

Assumption 2.3 (Analytic form factor)**.** *Let the form factor be rotation-symmetric $\phi(q)\equiv\phi(|q|)$, as in* (2.26)*, and let $\hat{\psi}$ be defined as in* (2.27)*. We assume that $\hat{\psi}(0)=0$ and that the function $\omega\to\hat{\psi}(\omega)$ has an analytic extension to a neighborhood of the strip $\mathbb{R}+\mathrm{i}[\delta_{\mathrm{R}},\delta_{\mathrm{R}}]$, for some $\delta_{\mathrm{R}}>0$, such that*

$$\sup_{-\delta_{\mathrm{R}}\leq\chi\leq\delta_{\mathrm{R}}}\int_{\mathbb{R}+\mathrm{i}\chi}\mathrm{d}\omega|\hat{\psi}(\omega)|<\infty. \tag{2.28}$$

We note that Assumption 2.3 is satisfied (in $d\geq 4$) if one chooses:

$$\phi(|q|) := \frac{1}{\sqrt{|q|}}\vartheta(|q|) \tag{2.29}$$

with ϑ a function on $\mathbb{R}$ with $\vartheta(-\omega)=\vartheta(\omega)$ and analytic in the strip of width δ_{R}, and such that (2.28) holds with $|\vartheta(\omega)|^2$ substituted for $|\hat{\psi}(\omega)|$.

The motivation for Assumptions 2.2 and 2.3 will become clear in Section 5.1, where we discuss the reservoir space-time correlation function $\psi(x,t)$.

The last assumption is a Fermi Golden Rule condition that ensures that the spin degrees of freedom are effectively coupled to the reservoir. To state it, we need the following operators

$$W_a := \sum_{\substack{e,e' \in \mathrm{sp} Y \\ e-e'=a}} 1_{e'}(Y) W 1_e(Y), \qquad a \in \mathrm{sp}(\mathrm{ad}(Y)) \tag{2.30}$$

Note that the variable a labels the Bohr-frequencies of the internal degrees of freedom of the particle.

Assumption 2.4 (Fermi Golden Rule)**.** *Recall the function $\hat{\psi}$ as defined in* (2.27)*. The set of matrices*

$$\mathscr{B}_{\mathrm{W}} := \left\{ \hat{\psi}(a) W_a, a \in \mathrm{sp}(\mathrm{ad}(Y)) \right\} \subset \mathscr{B}(\mathscr{S}) \tag{2.31}$$

generates the complete algebra $\mathscr{B}(\mathscr{S})$. This means that any $S \in \mathscr{B}(\mathscr{S})$ which commutes with all operators in $\mathscr{B}_{\mathrm{W}}$ is necessarily a multiple of the identity. We also require the following non-degeneracy condition

- *Every eigenvalue of Y is nondegenerate (multiplicity* 1*)*
- *For all eigenvalues e, e', e'', e''' of Y such that $e \neq e'$, we have*

$$e' - e = e''' - e'' \qquad \Rightarrow \qquad e' = e''' \text{ and } e'' = e \tag{2.32}$$

This condition implies that all eigenvalues of $\mathrm{ad}(Y)$ are nondegenerate, except for the eigenvalue 0*, whose multiplicity is given by $\dim \mathscr{S}$.*

The strict nondegeneracy condition on Y, in contrast to the condition on $\mathscr{B}_W$, is not crucial to our technique of proof, but it allows us to be more concrete in some stages of the calculation. In particular, the matrices $W_{a\neq 0}$, introduced above in (2.30), can be rewritten as

$$W_a = \langle e', We \rangle \times |e'\rangle\langle e|, \tag{2.33}$$

where e, e' are the unique eigenvalues s.t. $e - e' = a \neq 0$, and we denoted the corresponding eigenvectors by the same symbols e, e'. The condition that $\mathscr{B}_W$ generates the complete algebra, can than be rephrased as follows: Consider an undirected graph with vertex set $\mathrm{sp} Y$ and let the vertices e and e' be connected by an edge if and only if

$$\hat{\psi}(e'-e) |\langle e, We' \rangle|^2 \neq 0 \tag{2.34}$$

(note that this condition is indeed symmetric in e, e', as long as $\beta < \infty$). Then Assumption 2.4 is satisfied if and only if this graph is connected.

Assumptions of the type above have their origin in a criterion for ergodicity of quantum master equations due to [34, 19]. In our analysis, too, Assumption 2.4 is used to ensure that the Markovian semigroup Λ_t (to be introduced in Section 4) has good ergodic properties. This can be seen in Section C.1.1 in Appendix C.

2.4 The dynamics of the coupled system

Consider the Hilbert space $\mathscr{H} := \mathscr{H}_{\mathrm{S}} \otimes \mathscr{H}_{\mathrm{R}}$. The Hamiltonian H_λ (with coupling constant λ) on $\mathscr{H}$ is (formally) given by

$$H_\lambda := H_{\mathrm{S}} + H_{\mathrm{R}} + \lambda H_{\mathrm{SR}} \tag{2.35}$$

If the following condition is satisfied

$$\langle \phi, \omega^{-1} \phi \rangle_{\mathfrak{h}} < \infty, \tag{2.36}$$

then H_{SR} is a relatively bounded perturbation of $H_{\mathrm{S}} + H_{\mathrm{R}}$ and hence H_λ is a self-adjoint operator. One easily checks that (2.36) is implied by Assumptions 2.2 and 2.3.

For the purposes of our analysis, it is important to understand the dynamics of the coupled system at positive temperature. To this end, we introduce the reduced dynamics of the quantum particle.

By a slight abuse of notation, we use $\rho_{\mathrm{R}}^{\beta}$ to denote the conditional expectation $\mathscr{B}(\mathscr{H}_{\mathrm{S}}) \otimes \mathfrak{C} \to \mathscr{B}(\mathscr{H}_{\mathrm{S}})$, given by

$$\rho_{\mathrm{R}}^{\beta}(S \otimes R) = S \rho_{\mathrm{R}}^{\beta}(R) \tag{2.37}$$

where $\rho_{\mathrm{R}}^{\beta}(R)$ is defined by (2.21-2.22-2.23) for $R \in \mathfrak{C}$, i.e. a polynomial in creation and annihilation operators.

Formally, the reduced dynamics in the Heisenberg picture is given by

$$\mathcal{Z}_t^{\star}(S) := \rho_{\mathrm{R}}^{\beta}\left[\mathrm{e}^{\mathrm{i}tH_\lambda}\,(S \otimes 1)\,\mathrm{e}^{-\mathrm{i}tH_\lambda}\right]. \tag{2.38}$$

However, this definition does not make sense a priori, since $\mathrm{e}^{\mathrm{i}tH_\lambda}\,(S \otimes 1)\,\mathrm{e}^{-\mathrm{i}tH_\lambda} \notin \mathscr{B}(\mathscr{H}_{\mathrm{S}}) \otimes \mathfrak{C}$ in general. A mathematically precise definition of $\mathcal{Z}_t^{\star}$ is the subject of the upcoming Lemma 2.5.

Since both the initial reservoir state $\rho_{\mathrm{R}}^{\beta}$ and the Hamiltonian H_λ are translation-invariant, we expect that the reduced evolution $\mathcal{Z}_t^{\star}$ is also translation invariant in the sense that

$$\mathcal{T}_z \mathcal{Z}_t^{\star} \mathcal{T}_{-z} = \mathcal{Z}_t^{\star}, \qquad \text{where } (\mathcal{T}_z S)(x_L, x_R) := S(x_L + z, x_R + z) \tag{2.39}$$

By the requirement $\varepsilon(k) = \varepsilon(-k)$ in Assumption 2.1 and the requirement that $\phi(q) = \phi(-q)$ in Assumption 2.3, the Hamiltonian H_λ is also invariant with respect to space-inversion $x \mapsto -x$, or, equivalently, $k \mapsto -k$. Since the initial reservoir state is also invariant with respect to space inversion (this follows from the fact that $\omega(q) = \omega(-q)$), we expect that

$$\mathcal{T}_E \mathcal{Z}_t^{\star} \mathcal{T}_E^{-1} = \mathcal{Z}_t^{\star}, \qquad \text{where } (\mathcal{T}_E S)(x_L, x_R) := S(-x_L, -x_R) \tag{2.40}$$

Finally, the self-adjointness of the time-evolved density matrix implies that

$$\mathcal{T}_J \mathcal{Z}_t^{\star} \mathcal{T}_J^{-1} = \mathcal{Z}_t^{\star}, \qquad \text{where } (\mathcal{T}_J S)(x_L, x_R) := S^*(x_R, x_L) \tag{2.41}$$

where the * in $S^*(x_R, -x_L)$ is just the Hermitian adjoint of operators on $\mathscr{S}$.

Lemma 2.5. *Assume Assumptions 2.1, 2.2 and 2.3, and let*

$$H_0 := H_{\mathrm{S}} + H_{\mathrm{R}}, \qquad H_{\mathrm{SR}}(t) := \mathrm{e}^{\mathrm{i}tH_0} H_{\mathrm{SR}} \mathrm{e}^{-\mathrm{i}tH_0} \tag{2.42}$$

The Lie-Schwinger series

$$\begin{aligned} \mathcal{Z}_t^{\star}(S) \quad := \quad & \sum_{n \in \mathbb{N}} (\mathrm{i}\lambda)^n \int\limits_{0 \,\leq\, t_1 \,\leq\, \ldots \,\leq\, t_n \,\leq\, t} \mathrm{d}t_1 \ldots \mathrm{d}t_n \\ & \rho_{\mathrm{R}}^{\beta}\Big(\mathrm{ad}(H_{\mathrm{SR}}(t_1))\mathrm{ad}(H_{\mathrm{SR}}(t_2)) \ldots \mathrm{ad}(H_{\mathrm{SR}}(t_n))\,\mathrm{e}^{\mathrm{i}t\mathrm{ad}(H_0)}\,(S \otimes 1)\Big) \end{aligned} \tag{2.43}$$

is well-defined for all $\lambda, t \in \mathbb{R}$, that is, the RHS is a norm convergent family of operators and $\mathcal{Z}_t^{*}$ has the following properties

1) $\mathcal{Z}_t^{\star}(1) = 1$.
2) $\mathcal{T}_z \mathcal{Z}_t^{\star} \mathcal{T}_{-z} = \mathcal{Z}_t^{\star}$ *with $\mathcal{T}_z$ as defined in* (2.39).
3) $\mathcal{T}_E \mathcal{Z}_t^{\star} \mathcal{T}_E^{-1} = \mathcal{Z}_t^{\star}$ *with $\mathcal{T}_E$ as defined in* (2.40).
4) $\mathcal{T}_J \mathcal{Z}_t^{\star} \mathcal{T}_J^{-1} = \mathcal{Z}_t^{\star}$ *with $\mathcal{T}_J$ as defined in* (2.41).
5) $\|\mathcal{Z}_t^{\star}(S)\|_\infty \leq \|S\|_\infty$.
6) $\mathcal{Z}_t^{\star}(S) \geq 0$ *for* $S \geq 0$
7) *For $S \in \mathscr{B}_2(\mathscr{H}_{\mathrm{S}})$, the map $S \mapsto \mathcal{Z}_t^{\star}(S)$ is continuous in t in the Hilbert-Schmidt norm $\|\cdot\|_2$.*

These properties of $\mathcal{Z}_t^\star$ should not come as a surprise, they hold true trivially if one pretends that the initial reservoir state ρ_{R}^β is a density matrix and $\mathcal{Z}_t^\star$ is obtained by taking the partial trace over the reservoir space, as in (1.7). One can prove this lemma, under much less restrictive conditions than the stated assumptions, by estimates on the RHS. For this purpose, the estimates given in the present paper amply suffice. However, one can also define the system-reservoir dynamics as a dynamical system on a von Neumann algebra through the Araki-Woods representation and this is the usual approach in the mathematical physics literature, see e.g. [3, 12, 10, 20, 23].

We also define $\mathcal{Z}_t : \mathscr{B}_1(\mathscr{H}_{\mathrm{S}}) \to \mathscr{B}_1(\mathscr{H}_{\mathrm{S}})$, the reduced dynamics in the Schrödinger representation by duality

$$\mathrm{Tr}[S\mathcal{Z}_t^\star(S')] = \mathrm{Tr}[\mathcal{Z}_t(S)S'] \tag{2.44}$$

Physically, $\mathcal{Z}_t^\star$ is the reduced dynamics on observables of the system and $\mathcal{Z}_t$ is the reduced dynamics on states.

2.5 Translation invariance and the fiber decomposition

In this section, we introduce concepts and notation that will prove useful in the analysis of the reduced evolution $\mathcal{Z}_t$. These concepts will be used in Section 3.2. However, Section 3.1, which contains the main results, can be understood without the concepts introduced in the present section.

Consider the space of Hilbert-Schmidt operators

$$\mathscr{B}_2(\mathscr{H}_{\mathrm{S}}) \sim \mathscr{B}_2(l^2(\mathbb{Z}^d)\otimes\mathscr{S}) \sim L^2(\mathbb{T}^d\times\mathbb{T}^d, \mathscr{B}_2(\mathscr{S}), \mathrm{d}k_L\mathrm{d}k_R) \tag{2.45}$$

and define

$$\hat{S}(k_L,k_R) := \sum_{x_L,x_R\in\mathbb{Z}^d} S(x_L,x_R)\mathrm{e}^{-\mathrm{i}(x_Lk_L - x_Rk_R)}, \qquad S\in\mathscr{B}_2(l^2(\mathbb{Z}^d)\otimes\mathscr{S}). \tag{2.46}$$

Note the asymmetric normalization of the Fourier transform, which serves to eliminate factors of 2π in the bulk of the paper. In what follows, we will write S for $\hat{S}$ to keep the notation simple, since the arguments $x \leftrightarrow k$ will indicate whether we are dealing with S or $\hat{S}$. To deal conveniently with the translation invariance of our model, we change variables, see also Figure 1.

$$k = \frac{k_L+k_R}{2}, \qquad p = k_L - k_R, \qquad k,p\in\mathbb{T}^d \tag{2.47}$$

and, for a.e. $p\in\mathbb{T}^d$, we obtain a function $S_p\in L^2(\mathbb{T}^d,\mathscr{B}_2(\mathscr{S}))$ by putting

$$(S_p)(k) := S(k+\frac{p}{2}, k-\frac{p}{2}). \tag{2.48}$$

This follows from the fact that the Hilbert space $\mathscr{B}_2(\mathscr{H}_{\mathrm{S}}) \sim L^2(\mathbb{T}^d\times\mathbb{T}^d,\mathscr{B}_2(\mathscr{S}),\mathrm{d}k_L\mathrm{d}k_R)$ can be represented as a direct integral

$$\mathscr{B}_2(\mathscr{H}_{\mathrm{S}}) = \int_{\mathbb{T}^d}^{\oplus} \mathrm{d}p\, \mathscr{G}^p, \qquad S = \int_{\mathbb{T}^d}^{\oplus}\mathrm{d}p\, S_p, \tag{2.49}$$

where each 'fiber space' $\mathscr{G}^p$ is naturally identified with $\mathscr{G}\equiv L^2(\mathbb{T}^d,\mathscr{B}_2(\mathscr{S}))$. Elements of $\mathscr{G}$ will often be denoted by ξ,ξ' and the scalar product is

$$\langle\xi,\xi'\rangle_{\mathscr{G}} := \int_{\mathbb{T}^d}\mathrm{d}k\ \mathrm{Tr}_{\mathscr{S}}[\xi^*(k)\xi'(k)] \tag{2.50}$$

with $\mathrm{Tr}_{\mathscr{S}}$ the trace over the space of internal degrees of freedom $\mathscr{S}$.

Let $\mathcal{T}_z, z\in\mathbb{Z}^d$ be the lattice translation defined in (2.39). In momentum space,

$$(\mathcal{T}_zS)_p = \mathrm{e}^{-\mathrm{i}pz}S_p, \qquad S\in\mathscr{B}_2(\mathscr{H}_{\mathrm{S}}). \tag{2.51}$$

Since H_λ and ρ_{R}^β are translation invariant, it follows that

$$\mathcal{T}_{-z}\mathcal{Z}_t\mathcal{T}_z = \mathcal{Z}_t. \tag{2.52}$$

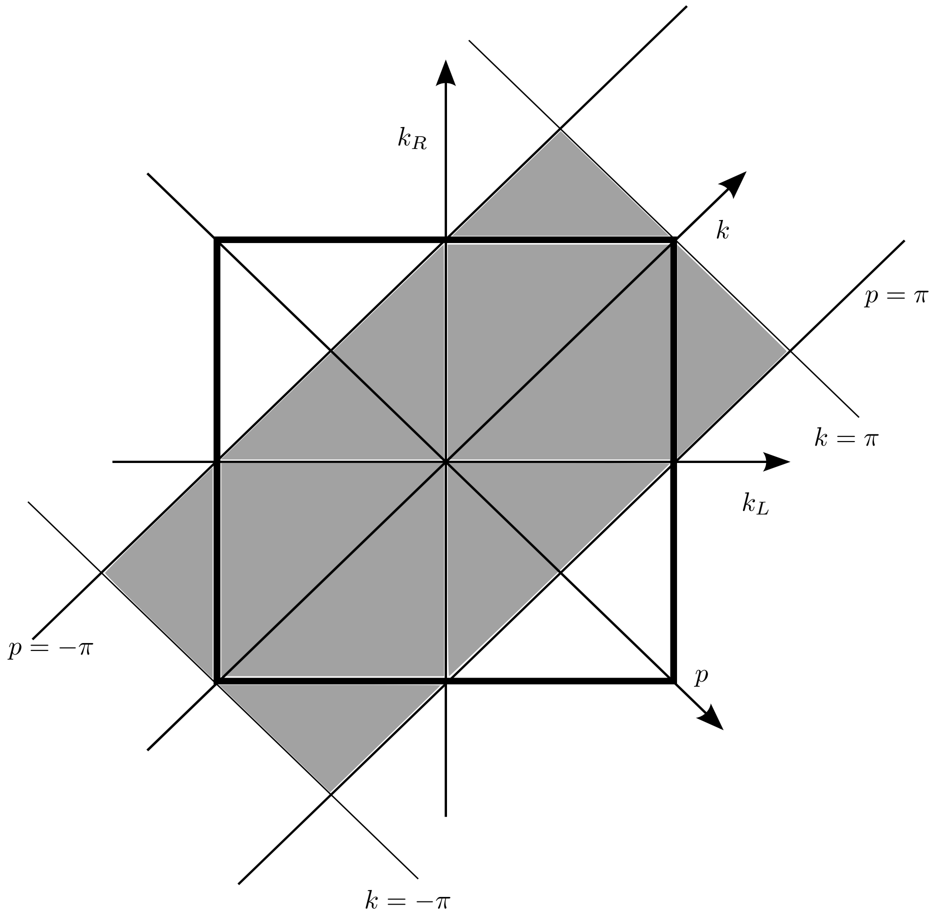


Figure 1: The thick black square $[-\pi, \pi] \times [-\pi, \pi]$ is the momentum space $\mathbb{T}^d \times \mathbb{T}^d$ (drawn here for $d = 1$), with $k_L, k_R \in \mathbb{T}^d$. After changing variables to $(k, p) \in \mathbb{T}^d \times \mathbb{T}^d$, the momentum space is transformed into the gray rectangle. On sees that the four triangles which lie inside the square but ouside the rectangle, are identified with the four triangles inside the rectangle but outside the square.

Let $\mathcal{W} \in \mathscr{B}(\mathscr{B}_2(\mathscr{H}_{\mathrm{S}}))$ be translation invariant in the sense that $\mathcal{T}_{-z}\mathcal{W}\mathcal{T}_z = \mathcal{W}$ (cf. (2.52)). Then it follows that, in the representation (2.49), $\mathcal{W}$ acts diagonally in p, i.e. $(\mathcal{W}S)_p$ depends only on S_p and we define $\mathcal{W}_p$ by

$$(\mathcal{W}S)_p = \mathcal{W}_p S_p, \qquad S_p \in \mathscr{G}, \mathcal{W}_p \in \mathscr{B}(\mathscr{G}) \tag{2.53}$$

For the sake of clarity, we give an explicit expression for $\mathcal{W}_p$. Define the kernel $\mathcal{W}_{x_L, x_R; x'_L, x'_R}$ by

$$(\mathcal{W}S)(x'_L, x'_R) = \sum_{x_L, x_R \in \mathbb{Z}^d} \mathcal{W}_{x_L, x_R; x'_L, x'_R} S(x_L, x_R), \qquad x'_L, x'_R \in \mathbb{Z}^d. \tag{2.54}$$

Translation invariance is expressed by

$$\mathcal{W}_{x_L, x_R; x'_L, x'_R} = \mathcal{W}_{x_L+z, x_R+z; x'_L+z, x'_R+z}, \qquad z \in \mathbb{Z}^d, \tag{2.55}$$

and, as an integral kernel, $\mathcal{W}_p \in \mathscr{B}(L^2(\mathbb{T}^d, \mathscr{B}_2(\mathscr{S})))$ is given by

$$\mathcal{W}_p(k', k) = \sum_{\substack{x_R, x'_L, x'_R \in \mathbb{Z}^d \\ x_L = 0}} \mathrm{e}^{\mathrm{i}k(x_L - x_R) - \mathrm{i}k'(x'_L - x'_R)} \mathrm{e}^{-\mathrm{i}\frac{p}{2}((x'_L + x'_R) - (x_L + x_R))} \mathcal{W}_{x_L, x_R; x'_L, x'_R}. \tag{2.56}$$

To avoid confusion with other subscripts we will often write

$$\{S\}_p \text{ instead of } S_p \qquad \text{and} \qquad \{\mathcal{W}\}_p \text{ instead of } \mathcal{W}_p \tag{2.57}$$

We also introduce the following transformations. For $\nu \in \mathbb{T}^d$, let U_ν be the unitary operator acting on the fiber spaces $\mathscr{G}$ as

$$(U_\nu \xi)(k) = \xi(k+\nu), \qquad \xi \in \mathscr{G} \tag{2.58}$$

Next, let $\kappa = (\kappa_L, \kappa_R) \in \mathbb{C}^d \times \mathbb{C}^d$ and define the operators $\mathcal{J}_\kappa$ by

$$(\mathcal{J}_\kappa S)(x_L, x_R) := \mathrm{e}^{\mathrm{i}\frac{1}{2}\kappa_L \cdot x_L} S(x_L, x_R) \mathrm{e}^{-\mathrm{i}\frac{1}{2}\kappa_R \cdot x_R} \tag{2.59}$$

Note that $\mathcal{J}_\kappa$ is unbounded if $\kappa \notin \mathbb{R}^d \times \mathbb{R}^d$.

The relation between the operators $\mathcal{J}_\kappa$ and the fiber decomposition is given by the relation

$$\{\mathcal{J}_\kappa \mathcal{W} \mathcal{J}_{-\kappa}\}_p \quad = \quad U_{-\frac{\kappa_L+\kappa_R}{4}} \{\mathcal{W}\}_{p-\frac{\kappa_L-\kappa_R}{2}} U_{\frac{\kappa_L+\kappa_R}{4}}, \tag{2.60}$$

as follows from (2.56) and the definition (2.59). From (2.56) and (2.60), we check that

$$p \qquad \text{is conjugate to} \qquad \frac{1}{2}\left((x'_L + x'_R) - (x_L + x_R)\right) \tag{2.61}$$

$$\nu \qquad \text{is conjugate to} \qquad (x_L - x_R) - (x'_L - x'_R). \tag{2.62}$$

We state an important lemma on the fiber decomposition.

Lemma 2.6. *Let $S \in \mathcal{B}_1(L^2(\mathbb{T}^d, \mathscr{S}))$. Then, S_p is well-defined, for every p, as a function in $L^1(\mathbb{T}^d, \mathscr{B}_2(\mathscr{S}))$ and*

$$\operatorname{Tr} \mathcal{J}_\kappa S = \sum_{x \in \mathbb{Z}^d} \mathrm{e}^{-\mathrm{i}px} S(x,x) = \langle 1, S_p \rangle_{\mathscr{G}}, \qquad \textit{with } p = -\frac{\kappa_L - \kappa_R}{2} \textit{ and } \kappa = (\kappa_L, \kappa_R) \tag{2.63}$$

where 1 stands for the constant function on $\mathbb{T}^d$ with value $1 \in \mathscr{B}(\mathscr{S})$. If, moreover, $\mathcal{J}_\kappa S$ is a Hilbert-Schmidt operator for $|\operatorname{Im} \kappa_{L,R}| \leq \delta'$, then the function

$$\mathbb{T}^d \mapsto \mathscr{G}: \qquad p \mapsto S_p, \tag{2.64}$$

as defined in (2.48)*, is well-defined for all $p \in \mathbb{T}^d$ and has a bounded-analytic extension to the strip $|\operatorname{Im} p| < \delta'$.*

The first statement of the lemma follows from the singular-value decomposition for trace-class operators. In fact, the correct statement asserts that one can *choose* S_p such that (2.63) holds. Indeed, one can change the value of the kernel $S(k_L, k_R)$ on the line $k_L - k_R = p$ without changing the operator S, and hence S_p in (2.63) can not be defined via (2.48) in general, if the only condition on S is $S \in \mathscr{B}_1$.

The second statement of Lemma 2.6 is the well-known relation between exponential decay of functions and analyticity of their Fourier transforms. Since we will always demand the initial density matrix ρ_0 to be such that $\|\mathcal{J}_\kappa \rho_0\|_2$ is finite for κ in a complex domain, we will mainly need the second statement of Lemma 2.6.

By employing Lemma 2.6 and the properties of $\mathcal{Z}_t^*$ listed in Lemma 2.5, it is easy to show that the function

$$k \mapsto \{\mathcal{Z}_t \rho_0\}_0 (k) \in \mathscr{B}(\mathscr{S}) \tag{2.65}$$

takes values in the positive matrices on $\mathscr{S}$ and is normalized, i.e.,

$$\int \mathrm{d}k \operatorname{Tr}_{\mathscr{S}}[\{\mathcal{Z}_t \rho_0\}_0 (k)] = \langle 1, \{\mathcal{Z}_t \rho_0\}_0 \rangle_{\mathscr{G}} = 1 \tag{2.66}$$

Further, the space-inversion symmetry and self-adjointness of the density matrix (the third and fourth property in Lemma 2.5) imply that

$$E \{\mathcal{Z}_t\}_p E = \{\mathcal{Z}_t\}_{-p}, \qquad \text{where } (E\xi)(k) := \xi(-k), \quad \text{for } \xi \in \mathscr{G}. \tag{2.67}$$

$$J \{\mathcal{Z}_t\}_p J = \{\mathcal{Z}_t\}_{-p}, \qquad \text{where } (J\xi)(k) := (\xi(k))^*, \quad \text{for } \xi \in \mathscr{G}. \tag{2.68}$$

where the * on $\xi(k)$ is the Hermitian conjugation on $\mathscr{B}(\mathscr{S})$.

3 Results

In this section, we describe our main results. In Section 3.1, we state the results in a direct way, emphasizing the physical phenomena. In Section 3.2, we describe more general statements that imply all the results stated in Section 3.1.

3.1 Diffusion, decoherence and equipartition

We choose the initial state of the particle to be a density matrix $\rho \in \mathscr{B}_1(\mathscr{H}_{\mathrm{S}})$ satisfying

$$\rho > 0, \qquad \mathrm{Tr}[\rho] = 1 \qquad \|\mathcal{J}_\kappa \rho\|_2 < \infty, \tag{3.1}$$

for κ in some neighborhood of $0 \in \mathbb{C}^d \times \mathbb{C}^d$. The condition $\|\mathcal{J}_\kappa \rho\|_2 < \infty$ reflects the fact that, at time $t = 0$, the particle is exponentially localized near the origin.

Our results describe the time-evolved density matrix $\rho_t := \mathcal{Z}_t \rho$. Note that ρ_t depends on λ, too. First, we state that the particle exhibits diffusive motion.

Define the probability density $\mu_t \equiv \mu_t^\lambda$, depending on the initial state $\rho \in \mathscr{B}_1(\mathscr{H}_{\mathrm{S}})$, by

$$\mu_t(x) := \mathrm{Tr}_{\mathscr{S}}\left[\rho_t(x,x)\right]. \tag{3.2}$$

It is easy to see that

$$\mu_t(x) \geq 0, \qquad \sum_{x \in \mathbb{Z}^d} \mu_t(x) = \mathrm{Tr}[\rho_t] = 1. \tag{3.3}$$

The following theorem states that the family of probability densities $\mu_t(\cdot)$ converges in distribution and in the sense of moments to a Gaussian, after rescaling space as $x \to \frac{x}{\sqrt{t}}$.

Theorem 3.1 (Diffusion)**.** *Assume Assumptions 2.1, 2.2, 2.3 and 2.4. Let the initial state ρ satisfy condition* (3.1) *and let μ_t be as defined in* (3.2).

There is a positive constant λ_0 such that, for $0 < |\lambda| \leq \lambda_0$,

$$\sum_{x \in \mathbb{Z}^d} \mu_t(x) \mathrm{e}^{-\frac{\mathrm{i}}{\sqrt{t}} q \cdot x} \qquad \underset{t \nearrow \infty}{\longrightarrow} \qquad \mathrm{e}^{-\frac{1}{2} q \cdot D_\lambda q} \tag{3.4}$$

with the diffusion matrix D_λ *given by*

$$D_\lambda = \lambda^2 (D_{rw} + o(\lambda)) \tag{3.5}$$

where D_{rw} is the diffusion matrix of the Markovian approximation to our model, to be defined in Section 4. Both D_λ and D_{rw} are strictly positive matrices (i.e., all eigenvalues are strictly positive) with real entries.

The convergence of $\mu_t(\cdot)$ to a Gaussian also holds in the sense of moments: For any natural number $\ell \in \mathbb{N}$, we have

$$(\nabla_q)^\ell \left(\sum_{x \in \mathbb{Z}^d} \mu_t(x) \mathrm{e}^{-\frac{\mathrm{i}}{\sqrt{t}} q \cdot x} \right) \qquad \underset{t \nearrow \infty}{\longrightarrow} \qquad (\nabla_q)^\ell \mathrm{e}^{-\frac{1}{2} q \cdot D_\lambda q}, \tag{3.6}$$

In particular, for $\ell = 2$, this means that

$$\frac{1}{t} \sum_{x \in \mathbb{Z}^d} x_i x_j \, \mu_t(x) \qquad \underset{t \nearrow \infty}{\longrightarrow} \qquad (D_\lambda)_{i,j} \tag{3.7}$$

Our next result describes the asymptotic 'state' of the particle. Not all observables reach a stationary value as $t \nearrow \infty$. For example, as stated in Theorem 3.1, the position diffuses. The asymptotic state applies to the internal degrees of freedom of the particle and to functions of its momentum. Hence, we look at observables of the form

$$F(P) \otimes A, \qquad F = \overline{F} \in L^\infty(\mathbb{T}^d), \ A = A^* \in \mathscr{B}(\mathscr{S}). \tag{3.8}$$

with $P = P \otimes 1$ the lattice momentum operator defined in Section 2.2. Such observables can be represented as elements of the Hilbert space $L^2(\mathbb{T}^d) \otimes \mathscr{B}_2(\mathscr{S}) \sim L^2(\mathbb{T}^d, \mathscr{B}_2(\mathscr{S})) = \mathscr{G}$ (recall that $\mathscr{S}$ is finite-dimensional) by the obvious mapping

$$F(P) \otimes A \mapsto F \otimes A \tag{3.9}$$

since $L^\infty(\mathbb{T}^d) \subset L^2(\mathbb{T}^d)$. Consequently, the asymptotic state is not described by a density matrix on $\mathscr{H}_{\mathrm{S}}$, but by a positive functional on the Hilbert space $\mathscr{G}$. This positive functional is called $\xi^{eq} \equiv \xi^{eq}_\lambda$ ('eq' for equilibrium) and we identify it with an element of $\mathscr{G}$. The asymptotic expectation value of $F \otimes A$ is given by

$$\langle F \otimes A, \xi^{eq} \rangle_{\mathscr{G}} = \int_{\mathbb{T}^d} \mathrm{d}k\, F(k)\, \mathrm{Tr}_{\mathscr{S}} \left[\xi^{eq}(k) A \right] \tag{3.10}$$

We also state a result on decoherence: Equation (3.13) expresses that the off-diagonal elements of ρ_t in position representation are exponentially damped in the distance from the diagonal. Note that this is not in contradiction with Theorem 3.1 as the latter speaks about diagonal elements of ρ_t.

Theorem 3.2 (Equipartition and decoherence)**.** *Assume Assumptions 2.1, 2.2, 2.3 and 2.4. Let the same conditions on the coupling constant λ and the initial state ρ be satisfied as in Theorem 3.1. Let A, F be as defined above. Then*

$$\mathrm{Tr}[\rho_t(F(P) \otimes A)] = \langle F \otimes A, \xi^{eq} \rangle_{\mathscr{G}} + O(\mathrm{e}^{-g\lambda^2 t}), \qquad t \nearrow \infty \tag{3.11}$$

for some decay rate $g > 0$. The function $\xi^{eq} \equiv \xi^{eq}_\lambda \in \mathscr{G}$ is given by

$$\xi^{eq}(k) = \frac{1}{Z(\beta)} \mathrm{e}^{-\beta Y} + o(|\lambda|^0), \qquad \textit{for all } k \in \mathbb{T}^d, \qquad \lambda \searrow 0 \tag{3.12}$$

with the normalization constant $Z(\beta) := (2\pi)^d\, \mathrm{Tr}(\mathrm{e}^{-\beta Y})$.

Further, there is a decoherence length $(\gamma_{dch})^{-1} > 0$ such that

$$\|\rho_t(x,y)\|_{\mathscr{B}(\mathscr{S})} \leq C \mathrm{e}^{-\gamma_{dch}|x-y|} + O(\mathrm{e}^{-g\lambda^2 t}), \qquad t \nearrow \infty \tag{3.13}$$

In particular, Theorem 3.2 implies that the inverse decoherence length γ_{dch} remains strictly positive as $\lambda \searrow 0$. Theorems 3.1 and 3.2 are derived from more general statements in the next section.

3.2 Asymptotic form of the reduced evolution

In the following theorem, we present a more general statement about the asymptotic form of the reduced evolution $\mathcal{Z}_t$. The two previous results, Theorems 3.1 and 3.2, are in fact immediate consequences of this more general statement.

As argued in Section 2.5 , the operator $\mathcal{Z}_t$ is translation invariant and hence it can be decomposed along the fibers,

$$\mathcal{Z}_t = \int_{\mathbb{T}^d}^{\oplus} \mathrm{d}p\, \{\mathcal{Z}_t\}_p\,, \qquad \{\mathcal{Z}_t\}_p \in \mathscr{B}(\mathscr{G}) \tag{3.14}$$

The next result, Theorem 3.3, lists some long-time properties of the operators $\{\mathcal{Z}_t\}$ and $U_\nu \{\mathcal{Z}_t\}_p U_{-\nu}$ with U_ν as defined in (2.58). To fix the domains of the parameters p and ν, we define

$$\mathfrak{D}^{low} \;:=\; \left\{ p \in \mathbb{T}^d + \mathrm{i}\mathbb{T}^d, \nu \in \mathbb{T}^d + \mathrm{i}\mathbb{T}^d \,\middle|\, |\mathrm{Re}\, p| < p^*,\ |\mathrm{Im}\, p| < \delta,\ |\mathrm{Im}\, \nu| < \delta \right\} \tag{3.15}$$

$$\mathfrak{D}^{high} \;:=\; \left\{ p \in \mathbb{T}^d + \mathrm{i}\mathbb{T}^d, \nu \in \mathbb{T}^d + \mathrm{i}\mathbb{T}^d \,\middle|\, |\mathrm{Re}\, p| > p^*/2,\ |\mathrm{Im}\, p| < \delta,\ |\mathrm{Im}\, \nu| < \delta \right\} \tag{3.16}$$

depending on some positive constants $p^*, \delta > 0$.

Theorem 3.3 (Asymptotic form of reduced evolution)**.** *Assume Assumptions 2.1, 2.2, 2.3 and 2.4, and let the same conditions on the coupling constant λ and the initial state ρ be satisfied as in Theorem 3.1. Then there are positive constants $p^* > 0$ and $\delta > 0$, determining the sets $\mathfrak{D}^{low}, \mathfrak{D}^{high}$ above, such that the following properties hold:*

1) *For small fibers p, i.e., such that $(p, 0) \in \mathfrak{D}^{low}$, there are rank-1 operators $P(p, \lambda)$, bounded operators $R^{low}(t, p, \lambda)$ and numbers $f(p, \lambda)$, analytic in p on $\mathfrak{D}^{low}$ and satisfying*

$$\sup_{(p,\nu)\in\mathfrak{D}^{low}} \|U_\nu P(p, \lambda) U_{-\nu}\| \quad < \quad C \tag{3.17}$$

$$\sup_{(p,\nu)\in\mathfrak{D}^{low}} \sup_{t\geq 0} \|U_\nu R^{low}(t, p, \lambda) U_{-\nu}\| \quad < \quad C \tag{3.18}$$

such that

$$\{\mathcal{Z}_t\}_p = \mathrm{e}^{f(p,\lambda)t} P(p, \lambda) + R^{low}(t, p, \lambda) \mathrm{e}^{-(\lambda^2 g^{low})t} \tag{3.19}$$

$$\sup_{(p,0)\in\mathfrak{D}^{low}} \operatorname{Re} f(p, \lambda) > -\lambda^2 g^{low} \tag{3.20}$$

for a positive rate $g^{low} > 0$.

2) *For large fibers p, i.e., such that $(p, 0) \in \mathfrak{D}^{high}$, there are bounded operators $R^{high}(t, p, \lambda)$, analytic in p on $\mathfrak{D}^{high}$ and satisfying*

$$\sup_{(p,\nu)\in\mathfrak{D}^{high}} \sup_{t\geq 0} \|U_\nu R^{high}(t, p, \lambda) U_{-\nu}\| = O(1), \qquad \lambda \searrow 0 \tag{3.21}$$

and

$$\{\mathcal{Z}_t\}_p = R^{high}(t, p, \lambda) \mathrm{e}^{-(\lambda^2 g^{high})t}, \qquad t \nearrow \infty \tag{3.22}$$

for some positive rate $g^{high} > 0$.

3) *The function $f(p, \lambda)$ and rank-1 operator $P(p, \lambda)$ satisfy*

$$\sup_{(p,0)\in\mathfrak{D}^{low}} \left| f(p, \lambda) - \lambda^2 f_{rw}(p) \right| \quad = \quad o(|\lambda|^2) \tag{3.23}$$

$$\sup_{(p,\nu)\in\mathfrak{D}^{low}} \|U_\nu P(p, \lambda) U_{-\nu} - U_\nu P_{rw}(p) U_{-\nu}\| \quad = \quad o(|\lambda|^0), \qquad \lambda \searrow 0 \tag{3.24}$$

where the function $f_{rw}(p)$ and the projection operator $P_{rw}(p)$ are defined in Section 4.

The main conclusion of this theorem is presented in Figure 2. Let $\mathcal{R}(z)$ be the Laplace transform of the reduced evolution $\mathcal{Z}_t$ and $\{\mathcal{R}(z)\}_p$ its fiber decomposition, i.e.,

$$\mathcal{R}(z) := \int_{\mathbb{R}^+} \mathrm{d}t\, \mathrm{e}^{-tz} \mathcal{Z}_t \qquad \text{and} \qquad \mathcal{R}(z) = \int_{\mathbb{T}^d}^{\oplus} \mathrm{d}p\, \{\mathcal{R}(z)\}_p \,. \tag{3.25}$$

The figure shows the singular points, $z = f(p, \lambda)$, of $\{\mathcal{R}(z)\}_p$. Those singular points determine the large time asymptotics. If we had not integrated out the reservoirs, i.e., if $\mathcal{Z}_t$ were the unitary dynamics, then one could identify $f(p, \lambda)$ with resonances of the generator of $\mathcal{Z}_t$.

The proof of Theorem 3.3 forms the bulk of the present paper.

3.3 Connection between Theorem 3.3 and the results in Section 3.1

In this section, we show how to derive Theorems 3.1 and 3.2 from Theorem 3.3.

Since $P(p, \lambda)$ is a rank-1 operator, we can write

$$P(p, \lambda) = \left|\xi(p, \lambda)\right\rangle\left\langle\tilde{\xi}(p, \lambda)\right|, \qquad \text{for some } \xi(p, \lambda), \tilde{\xi}(p, \lambda) \in \mathscr{G} \tag{3.26}$$

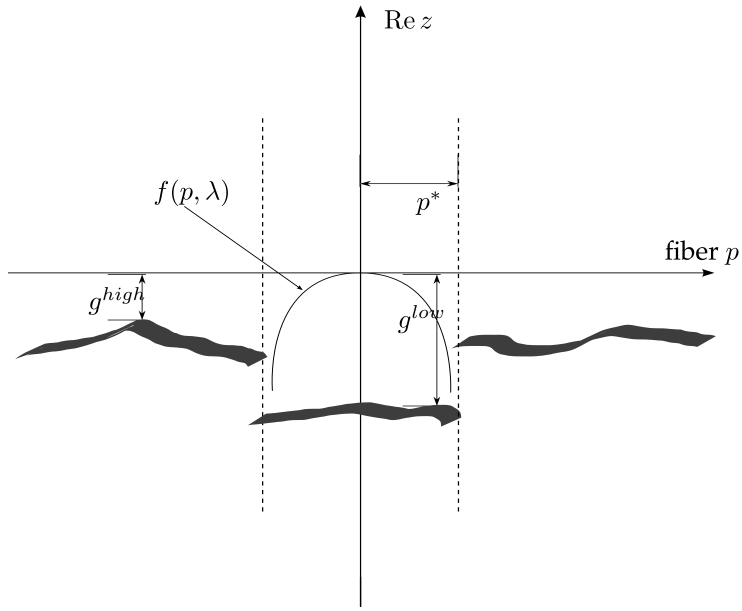


Figure 2: The singular points of $\{\mathcal{R}(z)\}_p$ as a function of the fiber momentum p. Above the irregular black line, the only singular points are given by $f(p, \lambda)$, in every small fiber p. Below the irregular black lines, we have no control.

using the notation introduced in (2.5). We derive a bound on the eigenvectors $\xi(p, \lambda)$ and $\tilde{\xi}(p, \lambda)$, analytically continued in the coordinate k. This bound follows from the analyticity and uniform boundedness of $U_\nu P(p, \lambda) U_{-\nu}$ on $\mathfrak{D}^{low}$ and straightforward symmetry arguments;

Lemma 3.4. *The vectors $U_\nu \xi(p, \lambda)$ and $U_\nu \tilde{\xi}(p, \lambda)$ can be chosen bounded-analytic on $\mathfrak{D}^{low}$. In other words, the operator $P(p, \lambda)$ has a kernel*

$$P(p, \lambda)(k, k') = \Big| \xi(p, \lambda)(k) \Big\rangle \Big\langle \tilde{\xi}(p, \lambda)(k') \Big| \tag{3.27}$$

which is bounded-analytic in both k and k' in the domain $|\mathrm{Im}\, k|, |\mathrm{Im}\, k'| < \delta$

Note that for fixed k, k', the RHS of (3.27) belongs to $\mathscr{B}(\mathscr{B}_2(\mathscr{S}))$.

Proof. Since $P(p, \lambda)$ is the dominant contribution to $\{\mathcal{Z}_t\}$ for large t, the symmetry properties of $\mathcal{Z}_t$, stated in Lemma 2.5, imply in particular that $(JE)P(p, \lambda)(JE)^{-1} = P(p, \lambda)$. Moreover, as JE is an involution, $JE = (JE)^{-1}$, the eigenvectors $\xi(p, \lambda)$ and $\tilde{\xi}(p, \lambda)$ can be chosen such that $JE\xi(p, \lambda) = \xi(p, \lambda)$ and $JE\tilde{\xi}(p, \lambda) = \tilde{\xi}(p, \lambda)$. Then

$$\|U_\nu \xi(p, \lambda)\| = \|U_\nu JE \xi(p, \lambda)\| = \|JEU_{-\nu} \xi(p, \lambda)\| = \|U_{-\nu} \xi(p, \lambda)\| \tag{3.28}$$

Since $U_\nu = \mathrm{e}^{-\mathrm{i}\nu \nabla_k}$, we have also

$$\|\xi(p, \lambda)\| \leq \|2 \cosh(\mathrm{Im}\, \nu \nabla_k) \xi(p, \lambda)\| = \|(U_{\mathrm{i Im}\, \nu} + U_{-\mathrm{i Im}\, \nu}) \xi(p, \lambda)\| \leq 2\|U_\nu \xi\|, \qquad \text{for any } \nu \in \mathbb{C}^d \tag{3.29}$$

The same relation holds for $\tilde{\xi}(p, \lambda)$ and hence none of the factors on the RHS of

$$\|U_\nu P(p, \lambda) U_{-\nu}\|_{\mathscr{B}(\mathscr{G})} = \Big\| \big| U_\nu \xi(p, \lambda) \big\rangle \big\langle U_\nu \tilde{\xi}(p, \lambda) \big| \Big\|_{\mathscr{B}(\mathscr{G})} = \|U_\nu \xi(p, \lambda)\|_{\mathscr{G}} \, \|U_\nu \tilde{\xi}(p, \lambda)\|_{\mathscr{G}} \tag{3.30}$$

can become small as ν varies. The lemma now follows from the uniform boundedness of $U_\nu P(p, \lambda) U_{-\nu}$. □

For $p=0$, the vectors $\xi(p,\lambda)$ and $\tilde{\xi}(p,\lambda)$ play a distinguished role, and we rename them as

$$\xi^{eq}=\xi^{eq}_{\lambda}:=\xi(p=0,\lambda), \qquad \tilde{\xi}^{eq}=\tilde{\xi}^{eq}_{\lambda}:=\tilde{\xi}(p=0,\lambda), \tag{3.31}$$

Note that ξ^{eq} was already referred to in Theorem 3.2.

By exploiting symmetry and positivity properties of the reduced evolution $\mathcal{Z}_t$, we can infer some further properties of the function $f(p,\lambda)$ and the operator $P(p,\lambda)$.

Proposition 3.5. *The function $f(p,\lambda)$, defined for all p with $(p,0)\in\mathfrak{D}^{low}$, has a negative real part, $\mathrm{Re}\, f(p,\lambda)\leq 0$, and satisfies the following properties*

$$f(p=0,\lambda)=0, \qquad \textit{and} \qquad \nabla_p f(p,\lambda)\big|_{p=0}=0 \tag{3.32}$$

$$\textit{The Hessian} \quad D_\lambda:=(\nabla_p)^2 f(p,\lambda)\big|_{p=0} \quad \textit{has real entries and is strictly positive} \tag{3.33}$$

The functions ξ^{eq} and $\tilde{\xi}^{eq}$ can be chosen such that

$$\tilde{\xi}^{eq}=1, \qquad \xi^{eq}(k)\geq 0, \qquad \int_{\mathbb{T}^d}\mathrm{d}k\,\mathrm{Tr}_{\mathscr{S}}\left[\xi^{eq}(k)\right]=\langle 1,\xi^{eq}\rangle=1 \tag{3.34}$$

where $1\in\mathscr{G}$ is the constant function on $\mathbb{T}^d$ with value $1\in\mathscr{B}_2(\mathscr{S})$. Moreover, it satisfies the space inversion symmetry $(\xi^{eq})(k)=(\xi^{eq})(-k)$.

The fact that $f(p=0,\lambda)=0$, $\tilde{\xi}^{eq}=1$ and (3.34) follow in a straightforward way from (2.66) and the asymptotic form (3.19). The symmetry property $\xi^{eq}(k)=\xi^{eq}(-k)$ and $\nabla_p f(p,\lambda)\big|_{p=0}=0$ follow from (2.67) and (3.19). The fact that D_λ has real entries follows from $f(p,\lambda)=\overline{f(-p,\lambda)}$ which in turn follows from the reality of the probabilities $\mu_t(x)$ and the convergence (3.4).

To derive the strict positivity of D_λ, we use the claim (in Proposition 4.2) that D_{rw}, the Hessian of $f_{rw}(p)$ at $p=0$, is strictly positive. By the convergence (3.23) and the analyticity of $f_{rw}(p)$, it follows that $|D_\lambda-\lambda^2 D_{rw}|\searrow 0$ as $\lambda\searrow 0$. Indeed, if a sequence of analytic functions is uniformly bounded on some open set and converges pointwise on that set, then all derivatives converge as well.

3.3.1 Diffusion

We outline the derivation of Theorem 3.1.

Let p be such that $(p,0)\in\mathfrak{D}^{low}$. Then we can calculate the logarithm of the characteristic function:

$$\begin{aligned}
\log\sum_x \mathrm{e}^{-\mathrm{i}px}\mu_t(x) &= \log\sum_x \mathrm{e}^{-\mathrm{i}px}\,\mathrm{Tr}_{\mathscr{S}}\,\rho_t(x,x)\\
&= \log\langle 1,\{\rho_t\}_p\rangle\\
&= \log\langle 1,\{\mathcal{Z}_t\}_p\{\rho_0\}_p\rangle\\
&= \log\left(\mathrm{e}^{f(p,\lambda)t}\langle 1,P(p,\lambda)\{\rho_0\}_p\rangle+\mathrm{e}^{-\lambda^2 g^{low}t}\langle 1,R^{low}(t,p,\lambda)\{\rho_0\}_p\rangle\right)\\
&= \log\mathrm{e}^{f(p,\lambda)t}\left(\langle 1,P(p,\lambda)\{\rho_0\}_p\rangle+\mathrm{e}^{-(\lambda^2 g^{low}-f(p,\lambda))t}C\|1\|\|\{\rho_0\}_p\|\right)\\
&= f(p,\lambda)t+\log\left(\langle 1,P(p,\lambda)\{\rho_0\}_p\rangle+\mathrm{e}^{-(\lambda^2 g^{low}-f(p,\lambda))t}C\|1\|\|\{\rho_0\}_p\|\right)
\end{aligned} \tag{3.35}$$

where the scalar product $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ refer to the Hilbert space $\mathscr{G}$. The second equality follows from Lemma 2.6, the fourth from (3.19) and the fifth from (3.18). The second term between brackets in the last line vanishes as $t\nearrow\infty$ by (3.20). To conclude the calculation, we need to check that the expression in $\log(\cdot)$ does not vanish. We note that

$$\langle 1,P(p=0,\lambda)\{\rho_0\}_0\rangle=\langle 1,\xi^{eq}\rangle\langle\tilde{\xi}^{eq},\{\rho_0\}_0\rangle=1 \tag{3.36}$$

as follows from the fact that $\tilde{\xi}^{eq}=1$ and the normalization of ξ^{eq} in (3.34). Hence, for p in a complex neighborhood of 0, the expression $\langle 1, P(p,\lambda)\left\{\rho_0\right\}_p\rangle$ is bounded away from 0 by analyticity in p. Consequently,

$$\lim_{t\nearrow\infty}\frac{1}{t}\log\sum_x \mathrm{e}^{-\mathrm{i}px}\mu_t(x) = f(p,\lambda). \tag{3.37}$$

Next, we remark that, for $\mathrm{i}p$ real, the LHS of (3.37) is a large deviation generating function for the family of measures $(\mu_t(\cdot))_{t\in\mathbb{R}^+}$. A classical result [5] in large deviation theory states that the analyticity of the large deviation generating function in a neighborhood of 0 implies a central limit theorem for the variable $\frac{x}{\sqrt{t}}$, both in distribution, see (3.4), as in the sense of moments, see (3.6).

3.3.2 Equipartition

To derive the result on equipartition in Theorem 3.2, we consider F, A as in (3.8). Since $\rho_t(F(P)\otimes A)$ is a trace-class operator, Lemma 2.6 implies that

$$\mathrm{Tr}\left[(F(P)\otimes A)\rho_t\right] = \langle 1, \{(F(P)\otimes A)\rho_t\}_0\rangle_{\mathscr{G}} = \langle F\otimes A, \{\rho_t\}_0\rangle_{\mathscr{G}} \tag{3.38}$$

where, as in (3.10), $F\otimes A$ stands for the function $k\mapsto F(k)A$ in $L^2(\mathbb{T}^d, \mathscr{B}_2(\mathscr{S}))$.

Using Theorem 3.3 for the fiber $p=0$, we obtain

$$\begin{aligned}\langle F\otimes A, \{\rho_t\}_0\rangle &= \mathrm{e}^{f(0,\lambda)t}\langle F\otimes A, P(p=0,\lambda)\{\rho_0\}_0\rangle + \mathrm{e}^{-(\lambda^2 g^{low})t}\langle F\otimes A, R^{low}(t,p=0,\lambda)\{\rho_0\}_0\rangle \\ &= \langle F\otimes A, \xi^{eq}\rangle + C\mathrm{e}^{-(\lambda^2 g^{low})t}\|F\otimes A\|_{\mathscr{G}}\|\{\rho_0\}_0\|_{\mathscr{G}}\end{aligned} \tag{3.39}$$

To obtain the second equality, we have used the uniform boundedness of the operators $R^{low}(t,p=0,\lambda)$ (Statement 1) of Theorem 3.3), the fact that $f(p=0,\lambda)=0$ (Proposition 3.5) and the identities

$$P(p=0,\lambda)\{\rho_0\}_0 = \langle\tilde{\xi}^{eq}, \{\rho_0\}_0\rangle\xi^{eq} = \langle 1, \{\rho_0\}_0\rangle\xi^{eq} = \xi^{eq} \tag{3.40}$$

Hence, from (3.39), we obtain the asymptotic expression (3.11) by choosing $g\leq g^{low}$.

3.3.3 Decoherence

In this section, we derive the bound (3.13) in Theorem 3.2. We decompose ρ_t as follows, using Theorem 3.3:

$$\rho_t := \int_{\mathbb{T}^d}^{\oplus} \mathrm{d}p\ \left\{\rho_t\right\}_p \tag{3.41}$$

$$= \underbrace{\int_{|p|\leq p^*}^{\oplus}\mathrm{d}p\,\mathrm{e}^{\lambda^2 f(p,\lambda)t}P(p,\lambda)\left\{\rho_0\right\}_p}_{=:A_1} + \mathrm{e}^{-\lambda^2 g^{low}t}\underbrace{\int_{|p|\leq p^*}^{\oplus}\mathrm{d}p\,R^{low}(t,p,\lambda)\left\{\rho_0\right\}_p}_{=:A_2} \tag{3.42}$$

$$+\ \mathrm{e}^{-\lambda^2 g^{high}t}\underbrace{\int_{|p|>p^*}^{\oplus}\mathrm{d}p\,R^{high}(t,p,\lambda)\left\{\rho_0\right\}_p}_{=:A_3} \tag{3.43}$$

$$\tag{3.44}$$

The terms A_2 and A_3 are bounded by

$$\|A_{2,3}\|_2^2 \leq C\int_{\mathbb{T}^d}\mathrm{d}p\|\left\{\rho_0\right\}_p\|_{\mathscr{G}}^2 = C\|\rho_0\|_2^2 \leq C\|\rho_0\|_1^2 \tag{3.45}$$

where the first inequality follows from the bounds (3.18) and (3.22). Hence, for our purposes, it suffices to consider the first term A_1. To calculate the operator A_1 in position representation, we use the kernel expression (3.27) for $P(p,\lambda)$ to obtain

$$(A_1)(x_L,x_R) = \int_{|p|\leq p^*} \mathrm{d}p \mathrm{e}^{\mathrm{i}\frac{p}{2}\cdot(x_L+x_R)} \mathrm{e}^{f(p,\lambda)t} \langle \tilde{\xi}(p,\lambda), \{\rho_0\}_p \rangle \int_{\mathbb{T}^d} \mathrm{d}k\, \xi(p,\lambda)(k) \mathrm{e}^{\mathrm{i}k\cdot(x_L-x_R)} \tag{3.46}$$

We now shift the path of integration (in k) into the complex plane, using that the function $\xi(p,\lambda)(\cdot)$ is bounded-analytic in a strip of width δ. This yields exponential decay in (x_L-x_R). Using also that $\operatorname{Re} f(p,\lambda)\leq 0$, for $|p|\leq p^*$ (see Proposition 3.5), we obtain the bound

$$\|(A_1)(x_L,x_R)\|_{\mathscr{B}_2(\mathscr{S})} \leq C\mathrm{e}^{-\gamma|x_L-x_R|}, \qquad \text{for } \gamma<\delta \tag{3.47}$$

Combining the bounds on A_1 and A_2, A_3, we obtain

$$\|\rho_t(x_L,x_R)\|_{\mathscr{B}_2(\mathscr{S})} \leq C\mathrm{e}^{-\gamma|x_L-x_R|} + C'\mathrm{e}^{-(\lambda^2 g)t}, \qquad \text{for } \gamma<\delta \tag{3.48}$$

with $g := \min\left(g^{low}, g^{high}\right)$. The fact that this bound is valid for any $\gamma<\delta$, confirms the claim that the inverse decoherence length γ_{dch} can be chosen uniformly in λ as $\lambda \searrow 0$.

4 The Markov approximation

For small coupling strength λ and times of order λ^{-2}, one can approximate the reduced evolution $\mathcal{Z}_t$ by a "quantum Markov semigroup" Λ_t which is of the form

$$\Lambda_t = \mathrm{e}^{t\left(-\mathrm{i}\,\mathrm{ad}(Y)+\lambda^2\mathcal{M}\right)} \tag{4.1}$$

where $Y = 1\otimes Y$ is the Hamiltonian of the internal degrees of freedom, and $\mathcal{M}$ is a Lindblad generator, see e.g. [1]. Lindblad generators, and especially the semigroups they generate, have received a lot of attention lately in quantum information theory. The operator $\mathcal{M}$ has the additional property of being translation-invariant. Translation-invariant Lindbladians have been classified in [21] and, recently, studied in a physical context; see [35] for a review. In Section 4.1, we construct $\mathcal{M}$ and we state its relation with $\mathcal{Z}_t$. We also describe heuristically how $\mathcal{M}$ emerges from time-dependent perturbation theory in λ as a lowest order approximation to $\mathcal{Z}_t$. In Section 4.2, we discuss the momentum representation of $\mathcal{M}$ (the derivation of this representation is however deferred to Appendix C), and we recognise that the evolution equation generated by $\mathcal{M}$ is a mixture of a linear Boltzmann equation for the translational degrees of freedom and a Pauli master equation for the internal degrees of freedom. In Section 4.3, we discuss spectral properties of $\mathcal{M}$, which are largely proven in Appendix C. Finally, in Section 4.3.1, we derive bounds on the long-time behavior of $\Lambda_t\rho$, for any density matrix $\rho\in\mathscr{B}_1(\mathscr{H}_\mathrm{S})$.

4.1 Construction of the semigroup

First, we define the operator $\hat{\mathcal{L}}^\star(t)$ on $\mathscr{B}(\mathscr{H}_\mathrm{S})$:

$$\hat{\mathcal{L}}^\star(t)(S) = -\rho_\mathrm{R}^\beta\left(\mathrm{ad}(H_\mathrm{SR})\,\mathrm{e}^{\mathrm{i}t\mathrm{ad}(Y+H_\mathrm{R})}\,\mathrm{ad}(H_\mathrm{SR})(S\otimes 1)\right) \tag{4.2}$$

This definition makes sense since the conditional expectation ρ_R^β is applied to an element of $\mathscr{B}(\mathscr{H}_\mathrm{S})\otimes\mathfrak{C}$, see Section 2.4. Then we consider the Laplace transform of $\hat{\mathcal{L}}^\star(t)$, i.e.

$$\mathcal{L}^\star(z) = \int_{\mathbb{R}^+} \mathrm{d}t\,\mathrm{e}^{-tz}\,\hat{\mathcal{L}}^\star(t), \qquad \operatorname{Re} z>0, \tag{4.3}$$

and, finally, we let $\mathcal{L}(z)$ be the dual operator to $\mathcal{L}^\star(z)$, acting on $\mathscr{B}_1(\mathscr{H}_\mathrm{S})$, see (2.44). Then the operator $\mathcal{M}$ is obtained from $\mathcal{L}$ by "spectral averaging" and adding the "Hamiltonian" term $-\mathrm{iad}(\varepsilon(P))$:

$$\mathcal{M} := -\mathrm{iad}(\varepsilon(P)) + \sum_{a\in \mathrm{sp}(\mathrm{ad}(Y))} 1_a(\mathrm{ad}(Y))\mathcal{L}(-\mathrm{i}a)1_a(\mathrm{ad}(Y)) \tag{4.4}$$

For now, this definition is formal, since it involves (4.3) with $\operatorname{Re} z = 0$.

The following proposition provides a careful definition of $\mathcal{M}$ and collects some basic properties of the semigroup evolution Λ_t.

Proposition 4.1. *Assume Assumptions 2.1, 2.2 and 2.3. Then, the operators $\mathcal{L}(z)$, defined above, can be continued from $\operatorname{Re} z > 0$ to a continuous function in the region $\operatorname{Re} z \geq 0$ and*

$$\sup_{\operatorname{Re} z\geq 0} \|\mathcal{J}_\kappa \mathcal{L}(z)\mathcal{J}_{-\kappa}\| < \infty, \qquad \textit{for } \kappa\in\mathbb{C}^d\times\mathbb{C}^d \tag{4.5}$$

(In fact, $\mathcal{J}_\kappa \mathcal{L}(z)\mathcal{J}_{-\kappa} = \mathcal{L}(z)$).

The operator $\mathcal{M}$, as defined in (4.4)*, is bounded both on $\mathscr{B}_1(\mathscr{H}_\mathrm{S})$ and $\mathscr{B}_2(\mathscr{H}_\mathrm{S})$. Recall the constants $q_\varepsilon(\gamma), \gamma>0$, defined in Assumption 2.1. Then*

$$\|\mathcal{J}_\kappa\mathcal{M}\mathcal{J}_{-\kappa} - \mathcal{M}\| \leq q_\varepsilon(|\operatorname{Im}\kappa_L|) + q_\varepsilon(|\operatorname{Im}\kappa_R|), \qquad |\operatorname{Im}\kappa_{L,R}| \leq \delta_\varepsilon \tag{4.6}$$

where the norm $\|\cdot\|$ refers to the operator norm on $\mathscr{B}(\mathscr{B}_2(\mathscr{H}_\mathrm{S}))$.

The family of operators Λ_t, defined in (4.1)*,*

$$\Lambda_t = \mathrm{e}^{t(-\mathrm{iad}(Y)+\lambda^2\mathcal{M})}, \qquad t\in\mathbb{R}^+ \tag{4.7}$$

is a "quantum dynamical semigroup". This means[3]*:*

$$\begin{array}{llll} i) & \Lambda_{t_1}\Lambda_{t_2} = \Lambda_{t_1+t_2} & \textit{for all } t_1,t_2\geq 0 & \textit{(semigroup property)} \\ ii) & \Lambda_t\rho \geq 0 & \textit{for any } 0\leq\rho\in\mathscr{B}_1(\mathscr{H}_\mathrm{S}) & \textit{(positivity preservation)} \\ iii) & \operatorname{Tr}\Lambda_t\rho = \operatorname{Tr}\rho & \textit{for any } 0\leq\rho\in\mathscr{B}_1(\mathscr{H}_\mathrm{S}) & \textit{(trace preservation)} \end{array} \tag{4.8}$$

We postpone the proof of this proposition to Appendix C.

4.1.1 Motivation of the semigroup Λ_t

The connection of the semigroup Λ_t with the reduced evolution $\mathcal{Z}_t$ is that, for any $T<\infty$,

$$\sup_{0<t<\lambda^{-2}T} \|\mathcal{Z}_t - \Lambda_t\|_{\mathscr{B}(\mathscr{B}_2(\mathscr{H}_\mathrm{S}))} = o(\lambda^0), \qquad \lambda\searrow 0 \tag{4.9}$$

Results in the spirit of (4.9) have been advocated by [22] and first proven, for confined (i.e. with no translational degrees of freedom) systems, in [9]. They go under the name "weak coupling limit" and they have given rise to extended mathematical studies, see e.g. [29, 11]. In our model, (4.9) will be implied by our proofs but we will not state it explicitly in the form given above. In fact, statements like (4.9) can be proven under much weaker assumptions than those in our model; see [33] for a proof which holds in all dimensions $d>1$.

Here, we restrict ourselves to a short and heuristic sketch of the way the Lindblad generator $\mathcal{M}$ emerges from the full dynamics. First, we consider the Lie-Schwinger series (2.43) in the interaction picture with respect to the free internal degrees of freedom, i.e. we consider $\mathcal{Z}_t^\star\mathrm{e}^{-\mathrm{i}t\mathrm{ad}(Y)}$ instead of $\mathcal{Z}_t^\star$. Keeping only terms up to second order in λ in (2.43) and substituting our definition for $\hat{\mathcal{L}}^\star(t)$ we obtain

$$\mathcal{Z}_t^\star\mathrm{e}^{-\mathrm{i}t\mathrm{ad}(Y)} \;=\; 1 + \mathrm{i}\lambda^2\, t\mathrm{ad}(\varepsilon(P)) + \lambda^2 \int\limits_{0<t_1<t_2<t} \mathrm{d}t_1\mathrm{d}t_2\, \mathrm{e}^{\mathrm{i}(t-t_2)\mathrm{ad}(Y)}\hat{\mathcal{L}}^\star(t_2-t_1)\mathrm{e}^{-\mathrm{i}(t-t_1)\mathrm{ad}(Y)} + O(\lambda^4) \tag{4.10}$$

[3]Most authors include "complete positivity" as a property of quantum dynamical semigroups, see e.g. [1]. Although the operators Λ_t satisfy complete positivity, we do not stress this since it is not important for our analysis.

where we have also used $[\varepsilon(P), Y] = 0$ to simplify the second term on the RHS. It is useful to rewrite the third term by splitting $t_2 = t_1 + (t_2 - t_1)$ and inserting the spectral decomposition of unity corresponding to the operator $\mathrm{ad}(Y)$

$$\sum_{a,a' \in \mathrm{sp}(\mathrm{ad}(Y))} \int_0^t \mathrm{d}t_1 \, \mathrm{e}^{\mathrm{i}(t-t_1)(a-a')} \, 1_a(\mathrm{ad}(Y)) \left(\int_0^{t-t_1} \mathrm{d}u \, \mathrm{e}^{-iua} \mathcal{L}^\star(u) \right) 1_{a'}(\mathrm{ad}(Y)) \tag{4.11}$$

Next, we analyze the RHS of (4.10) for long times; we choose $t = \lambda^{-2}\mathfrak{t}$, and we argue that, in the limit $\lambda \searrow 0$, the three terms reduce to the operator

$$1 + \mathrm{i}\mathrm{ad}(\varepsilon(P))\mathfrak{t} + \left(\sum_a 1_a(\mathrm{ad}(Y)) \mathcal{L}^\star(\mathrm{i}a) 1_a(\mathrm{ad}(Y)) \right) \mathfrak{t} \tag{4.12}$$

This limit can be straightforwardly justified if the function $t \to \mathcal{L}^*(t)$ is (norm-)integrable, which will follow from Assumptions 2.2 and 2.3 in our case. Indeed, if $t \to \mathcal{L}^*(t)$ is integrable, then the integral $\int_0^{\lambda^{-2}\mathfrak{t}-t_1} \mathrm{d}u \ldots$ in (4.11) converges to $\int_0^\infty \mathrm{d}u \ldots$ for fixed t_1, as $\lambda \to 0$. This yields the Laplace transform $\mathcal{L}^\star(\mathrm{i}a)$. The restriction $a = a'$ appears then because

$$\lambda^2 \int_0^{\lambda^{-2}\mathfrak{t}} \mathrm{d}t_1 \, \mathrm{e}^{\mathrm{i}(t-t_1)(a-a')} \quad \underset{\lambda \to 0}{\to} \quad \mathfrak{t}\delta_{a,a'} \tag{4.13}$$

and one finishes the argument by invoking dominated convergence. Comparing with (4.4) and using that $(1_a(\mathrm{ad}(Y)))^\star = 1_{-a}(\mathrm{ad}(Y))$, one checks that (4.12) is equal to $1 + \mathfrak{t}\mathcal{M}^\star$, with $\mathcal{M}^\star$ the dual to $\mathcal{M}$. This is the beginning of a series defining the semigroup $\mathrm{e}^{\mathfrak{t}\mathcal{M}^\star}$ (we got only the first two terms because we kept only terms of order λ^0 and λ^2 in the original Lie-Schwinger series).

4.2 Momentum space representation of $\mathcal{M}$

In this section, we give an explicit and intuitive expression for the operator $\mathcal{M}$. As $\mathcal{M}$ is translation covariant, i.e., $\mathcal{T}_z \mathcal{M} \mathcal{T}_{-z} = \mathcal{M}$, as in (2.52), we have the fiber decomposition,

$$\mathcal{M} = \int_{\mathbb{T}^d}^{\oplus} \mathrm{d}p \, \mathcal{M}_p \tag{4.14}$$

where the notation is as introduced in Section 2.5. We describe $\mathcal{M}_p$ explicitly as an operator on $\mathscr{G}$. It is of the form

$$(\mathcal{M}_p \xi)(k) = -\mathrm{i}[\Upsilon, \xi(k)] - \mathrm{i}(\varepsilon(k + \frac{p}{2}) - \varepsilon(k - \frac{p}{2}))\xi(k) + (\mathcal{N}\xi)(k), \qquad \xi \in \mathscr{G} \tag{4.15}$$

where ε is the dispersion law of the particle, see Section 2.2, and Υ is a self-adjoint matrix in $\mathscr{B}(\mathscr{S})$ whose only relevant property is that it commutes with Y, i.e., $[Y, \Upsilon] = 0$. Physically, it describes the Lamb-shift of the internal degrees of freedom due to the coupling to the reservoir and its explicit form is given in Appendix C. The operator $\mathcal{N}$ is given, for $\xi \in \mathcal{C}(\mathbb{T}^d, \mathscr{B}_2(\mathscr{S}))$, by

$$(\mathcal{N}\xi)(k) = \sum_{a \in \mathrm{sp}(\mathrm{ad}(Y))} \int \mathrm{d}k' \left(r_a(k', k) W_a \xi(k') W_a^* - \frac{1}{2} r_a(k, k') \left(\xi(k) W_a^* W_a + W_a^* W_a \xi(k) \right) \right) \tag{4.16}$$

with the (singular) jump rates

$$r_a(k, k') := 2\pi \int_{\mathbb{R}^d} \mathrm{d}q \, |\phi(q)|^2 \begin{cases} \frac{1}{1-\mathrm{e}^{-\beta\omega(q)}} \delta(\omega(q) - a) \delta_{\mathbb{T}^d}(k - k' - q) & a \geq 0 \\ \frac{1}{\mathrm{e}^{\beta\omega(q)}-1} \delta(\omega(q) + a) \delta_{\mathbb{T}^d}(k + q - k') & a < 0 \end{cases} \tag{4.17}$$

where ϕ is the form-factor, see Section 2.3, and $\delta_{\mathbb{T}^d}(\cdot)$ is the Dirac delta distribution on the torus, i.e.;

$$\delta_{\mathbb{T}^d}(\cdot) := \sum_{q_0 = 0 + (2\pi\mathbb{Z})^d} \delta(\cdot - q_0), \tag{4.18}$$

Note that $r_a(k,k')$ vanishes at $a=0$, due to the fact that the 'efficitive squared form factor' $\hat{\psi}(\cdot)$ vanishes at 0, see Assumption 2.3.

Equation (4.15) is most easily checked starting from the expressions for $\mathcal{M}$ in Section C.1. In particular, the three terms in (4.16) correspond to the fiber decompositions of the operators $\Phi(\rho), -\frac{1}{2}\Phi^\star(1)\rho, -\frac{1}{2}\rho\Phi^\star(1)$ in (C.7), and the first two terms on the RHS of (4.15) correspond to the commutator with Υ and $\varepsilon(P)$ in (C.7).

We already stated that $\mathcal{M}$ is translation-invariant, hence it commutes with $\mathrm{ad}(P)$. However, the operator $\mathcal{M}$ also commutes with $\mathrm{ad}(Y)$, as can be easily checked starting from the expressions (4.15) and (4.16) and employing the definitions of W_a in (2.30) and the fact that $[Y,\Upsilon]=0$.

We can therefore construct the double decomposition

$$\mathcal{M} = \underset{a\in\mathrm{sp}(\mathrm{ad}(Y))}{\oplus} \int_{\mathbb{T}^d}^{\oplus} \mathrm{d}p\, \mathcal{M}_{p,a} \tag{4.19}$$

where

$$\mathcal{M}_{p,a} := 1_a(\mathrm{ad}(Y))\mathcal{M}_p 1_a(\mathrm{ad}(Y)) \tag{4.20}$$

To proceed, we make use of our strong nondegeneracy condition in Assumption 2.4. Indeed, the operators $\mathcal{M}_{p,a}$ act on functions $\xi\in\mathscr{G}$ that satisfy the constraint

$$\xi(k) = 1_a(\mathrm{ad}(Y))\xi(k) = \sum_{e,e'\in\mathrm{sp}Y, e-e'=a} 1_e(Y)\xi(k)1_{e'}(Y), \qquad \xi\in\mathscr{G}\sim L^2(\mathbb{T}^d,\mathscr{B}_2(\mathscr{S})) \tag{4.21}$$

Due to the non-degeneracy assumption, the sum on the RHS contains only one non-zero term for $a\neq 0$, i.e., there are unique eigenvalues e,e' such that $a=e-e'$. Let us denote the eigenvector in the space $\mathscr{S}$ of the operator Y with eigenvalue e by e as well (cfr. the discussion following Assumption 2.4) , then this unique term in (4.21) can be written as

$$1_e(Y)\xi(k)1_{e'}(Y) = (\langle e,\xi(k)e'\rangle)\ |e\rangle\langle e'|, \qquad e-e'=a \tag{4.22}$$

It follows that the matrix valued function $\xi(k)$ satisfying (4.21) can be identified with the $\mathbb{C}$-valued function

$$\varphi(k) \equiv \langle e,\xi(k)e'\rangle_{\mathscr{S}} \tag{4.23}$$

For $a=0$, a function $\xi(k)$ satisfying (4.21) is necessarily diagonal in the basis of eigenvectors of Y. In that case, we can identify ξ with

$$\varphi(k,e) \equiv \langle e,\rho(k,k)e\rangle_{\mathscr{S}} \tag{4.24}$$

Hence, we can identify $\mathcal{M}_{p,a\neq 0}$ with an operator on $L^2(\mathbb{T}^d)$ and $\mathcal{M}_{p,0}$ with an operator on $L^2(\mathbb{T}^d\times\mathrm{sp}Y)$.

A careful analysis of these operators is performed in Appendix C. Here, we discuss the operator $\mathcal{M}_{0,0}$ because it is crucial for understanding our model.

4.2.1 The Markov generator $\mathcal{M}_{0,0}$

Let us choose $\varphi\in\mathcal{C}(\mathbb{T}^d\times\mathrm{sp}Y)$. Then, by the formulas given above, the operator $\mathcal{M}_{0,0}$ acts as

$$\mathcal{M}_{0,0}\varphi(k,e) := \int_{\mathbb{T}^d}\mathrm{d}k' \sum_{e'\in\mathrm{sp}Y} \left(r(k',e';k,e)\varphi(k',e') - r(k,e;k',e')\varphi(k,e)\right) \tag{4.25}$$

where $r(k,e;k',e')$ are (singular) *transition rates* given explicitly by

$$r(k,e;k',e') := r_{e-e'}(k,k')|\langle e',We\rangle|^2 \tag{4.26}$$

In formula (4.25), one recognizes the structure of a Markov generator, acting on absolutely continuous probability measures (hence L^1-functions) on $\mathbb{T}^d\times\mathrm{sp}Y$. The numbers

$$j(e,k) := \int_{\mathbb{T}^d}\mathrm{d}k' \sum_{e'\in\mathrm{sp}Y} r(k,e;k',e') \tag{4.27}$$

are called *escape rates* in the context of Markov processes. Let $\|\varphi\|_1 = \int_{\mathbb{T}^d} \mathrm{d}k \sum_{e \in \mathrm{sp}Y} |\varphi(k,e)|$. Then one easily checks by decomposing $\varphi = \varphi_+ + \varphi_-$ and using that $r(k,e;k',e')\mathrm{d}k'$ is a positive measure, that

$$\|\mathcal{M}_{0,0}\varphi\|_1 \leq 4 \left(\sup_{e,k} j(e,k) \right) \|\varphi\|_1, \tag{4.28}$$

which implies that $\mathcal{M}_{0,0}$ is bounded on $L^1(\mathbb{T}^d \times \mathrm{sp}Y)$. In particular, this means that $\mathcal{M}_{0,0}$ is a bonafide Markov generator (i.e. it generates a strongly continuous (in our case even norm-continuous) semigroup) and $\mathrm{e}^{t\mathcal{M}_{0,0}}\varphi$ is a probability measure for all $t \geq 0$. Physically speaking, the probability density φ is read off from the diagonal part of the density matrix ρ, see (4.24).

It is interesting to see that the transition rates $r(k,e;k',e')$ satisfy detailed balance at inverse temperature β for the internal energy levels e, e', and at infinite temperature for the momenta k, k'.

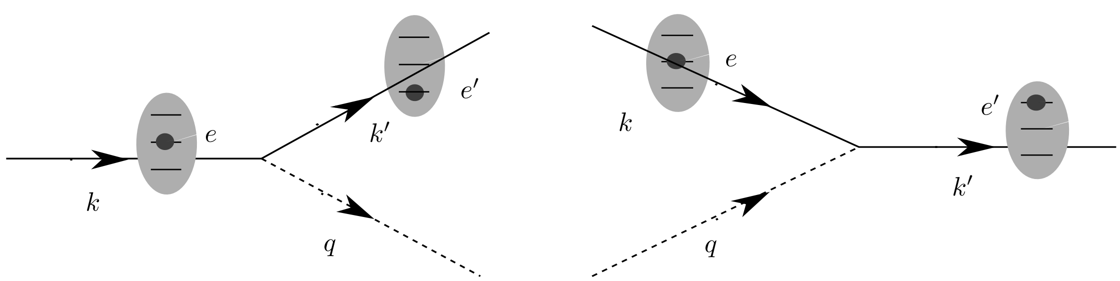
$$r(k,e;k',e') = \mathrm{e}^{\beta(e-e')} r(k,' e'; k, e) \tag{4.29}$$

Physically, we would expect overall detailed balance at inverse temperature β, i.e.

$$r(k,e;k',e') = \mathrm{e}^{\beta(E(e,k)-E(e',k'))} r(k,' e'; k, e) \tag{4.30}$$

where the energy $E(k,e)$ should depend on both e and k. To understand why E does not depend on k in (4.29), we recall that the kinetic energy of the particle is assumed to be of order λ^2; hence, the total energy is $e + \lambda^2 \varepsilon(k)$ which reduces to e in zeroth order in λ.

One can associate an intuitive picture with the operator $\mathcal{M}_{0,0}$. It describes the stochastic evolution of a particle with momentum k and energy e. The state of the particle changes from (k,e) to (k',e') by emitting and absorbing reservoir particles with momentum q and energy $\omega(q)$, such that total momentum and total energy (which does not include any contribution from k, k') are conserved, see Figure 3.



(a) Emission of a boson, $k = k' + q$ and $e = \omega(q) + e'$. (b) Absorption of a boson, $k + q = k'$ and $\omega(q) + e = e'$.

Figure 3: The processes contributing to the gain term (the first term on the RHS in (4.25)) of the operator $\mathcal{M}_{0,0}$. Emission corresponds to $e > e'$ and absorption to $e < e'$.

It is clear from the collision rules in Figure 3 that, in the absence of internal degrees of freedom, the particle can only emit or absorb bosons with momentum $q = 0$, and hence it cannot change its momentum. This means that without the internal degrees of freedom, the semigroup Λ_t would not exhibit any diffusive motion. This is indeed the reason why we introduced these internal degrees of freedom.

4.3 Asymptotic properties of the semigroup

The following Proposition 4.2 states some spectral results on the Lindblad operator $\mathcal{M}$ and its restriction to momentum fibers $\mathcal{M}_p \in \mathscr{B}(\mathscr{G})$. These results are stated in a way that mirrors, as closely as possible, the statements of Theorem 3.3.

These results are useful for two purposes. First of all, they show that our main physical results, Theorems 3.1 and 3.2, hold true if one replaces the reduced evolution $\mathcal{Z}_t$ by the semigroup Λ_t (see the remark following

Proposition 4.2). Second, a bound which follows directly from Proposition 4.2 will be a crucial ingredient in the proof of our main result Theorem 3.3. This bound is stated in (4.55) in Section 4.3.1.

We introduce the following sets (cf. (3.15-3.16))

$$\mathfrak{D}_{rw}^{low} := \left\{ p \in \mathbb{T}^d + \mathrm{i}\mathbb{T}^d, \nu \in \mathbb{T}^d + \mathrm{i}\mathbb{T}^d \middle| \, |\mathrm{Re}\, p| < p_{rw}^*, \ |\mathrm{Im}\, p| \leq \delta_{rw}, \ |\mathrm{Im}\, \nu| \leq \delta_{rw} \right\} \tag{4.31}$$

$$\mathfrak{D}_{rw}^{high} := \left\{ p \in \mathbb{T}^d + \mathrm{i}\mathbb{T}^d, \nu \in \mathbb{T}^d + \mathrm{i}\mathbb{T}^d \middle| \, |\mathrm{Re}\, p| > \frac{1}{2} p_{rw}^*, \ |\mathrm{Im}\, p| \leq \delta_{rw}, \ |\mathrm{Im}\, \nu| \leq \delta_{rw} \right\}, \tag{4.32}$$

depending on positive parameters $p_{rw}^* > 0$ and $\delta_{rw} > 0$. The subscript 'rw' stands for 'random walk' and it will always be used for objects related to Λ_t.

Proposition 4.2. *Assume Assumptions 2.1, 2.2, 2.3 and 2.4. There are positive constants $p_{rw}^* > 0$ and $\delta_{rw} > 0$, determining $\mathfrak{D}_{rw}^{low}, \mathfrak{D}_{rw}^{high}$ above, such that the following properties hold*

1) *For small fibers p, i.e., such that $(p, \nu) \in \mathfrak{D}_{rw}^{low}$, the operator $U_\nu \mathcal{M}_p U_{-\nu}$ is bounded and has a simple eigenvalue $f_{rw}(p)$, independent of ν,*
$$\mathrm{sp}(U_\nu \mathcal{M}_p U_{-\nu}) = \{f_{rw}(p)\} \cup \Omega_{p,\nu} \tag{4.33}$$
The eigenvalue $f_{rw}(p)$ is elevated above the rest of the spectrum, uniformly in p, i.e., there is a positive $g_{rw}^{low} > 0$ such that
$$\mathrm{Re}\, \Omega_{p,\nu} < -g_{rw}^{low} < \mathrm{Re}\, f_{rw}(p) \leq 0, \qquad \textit{for all } (p,\nu) \in \mathfrak{D}_{rw}^{low} \tag{4.34}$$
The one-dimensional spectral projector $U_\nu P_{rw}(p) U_{-\nu}$ corresponding to the eigenvalue $f_{rw}(p)$ satisfies
$$\sup_{(p,\nu)\in \mathfrak{D}^{low}} \| U_\nu P_{rw}(p) U_{-\nu} \| \leq C \tag{4.35}$$

2) *For large fibers p, i.e., such that $(p, 0) \in \mathfrak{D}_{rw}^{high}$, the operator $U_\nu \mathcal{M}_p U_{-\nu}$ is bounded and its spectrum lies entirely below the real axis, i.e.,*
$$\sup_{(p,\nu)\in \mathfrak{D}_{rw}^{high}} \mathrm{Re}\, \mathrm{sp}\, (U_\nu \mathcal{M}_p U_{-\nu}) < -g_{rw}^{high}, \qquad \textit{for some } g_{rw}^{high} > 0 \tag{4.36}$$

3) *The function $f_{rw}(p)$, defined for all p such that $(p, 0) \in \mathfrak{D}_{rw}^{low}$, has a negative real part, $\mathrm{Re}\, f_{rw}(p) \leq 0$, and satisfies*
$$f_{rw}(p=0) = 0, \qquad \textit{and} \qquad \nabla_p f_{rw}(p))\big|_{p=0} = 0 \tag{4.37}$$
$$\textit{The Hessian} \quad D_{rw} := (\nabla_p)^2 f_{rw}(p)\big|_{p=0} \quad \textit{has real entries and is strictly positive} \tag{4.38}$$
The spectral projector $P_{rw}(p=0)$ is given by
$$P_{rw}(p=0) = |\tilde{\xi}_{rw}^{eq}\rangle\langle \xi_{rw}^{eq}|, \tag{4.39}$$
with
$$\tilde{\xi}_{rw}^{eq}(k) = 1_{\mathscr{B}(\mathscr{S})}, \qquad \textit{and} \qquad \xi_{rw}^{eq}(k) = \frac{1}{(2\pi)^d} \frac{\mathrm{e}^{-\beta Y}}{\mathrm{Tr}(\mathrm{e}^{-\beta Y})}, \qquad k \in \mathbb{T}^d \tag{4.40}$$

The conclusion of Proposition 4.2 is sketched in Figure 4. The proof of this proposition is very analogous to [7] (which, however, does not consider internal degrees of freedom). For completeness, we reproduce the proof in Appendix C.

From Proposition 4.2, one can derive that the semigroup $\mathrm{e}^{t(-\mathrm{iad}(Y)+\mathcal{M})}$ exhibits diffusion, decoherence and equipartition. This follows by analogous reasoning as in Sections 3.3.1, 3.3.2 and 3.3.3, but starting from Proposition 4.2 instead of Theorem 3.3. The matrix D_{rw} is the diffusion constant, the inverse decoherence length has to be chosen smaller than δ_{rw} and the function ξ_{rw}^{eq} is the 'equilibrium state'. We do not derive these properties explicitly as they are not necessary for the proof of our main results.

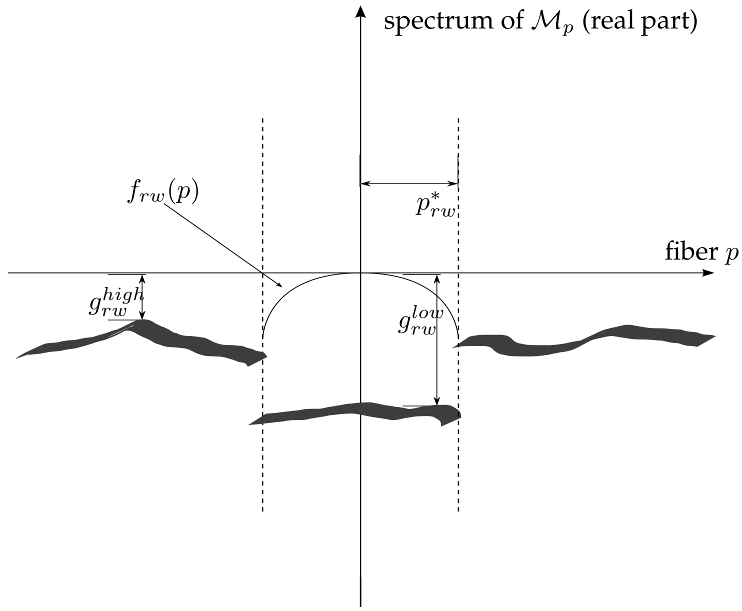


Figure 4: The spectrum of $\mathcal{M}_p$ as a function of the fiber momentum p. Above the irregular black line, the only spectrum consists of the isolated eigenvalue $f_{rw}(p)$, in every small fiber p. Below the irregular black lines, we have no control.

4.3.1 Bound on Λ_t in position representation

By virtue of Proposition 4.2, we can write

$$\{\Lambda_t\}_p \;=\; P_{rw}(p)\mathrm{e}^{\lambda^2 f_{rw}(p)t} + R^{low}_{rw}(t,p)\mathrm{e}^{-\lambda^2 g^{low}_{rw} t}, \qquad (p,\nu)\in\mathfrak{D}^{low}_{rw} \tag{4.41}$$

$$\{\Lambda_t\}_p \;=\; R^{high}_{rw}(p,t)\mathrm{e}^{-\lambda^2 g^{high}_{rw} t}, \qquad (p,\nu)\in\mathfrak{D}^{high}_{rw}, \tag{4.42}$$

with $P_{rw}(p)$ as defined above, satisfying (4.35), and the operators $R^{low}_{rw}, R^{high}_{rw}$ satisfying

$$\sup_{(p,\nu)\in\mathfrak{D}^{low}_{rw}} \sup_{t\geq 0} \|U_\nu R^{low}_{rw}(t,p)U_{-\nu}\| < C \tag{4.43}$$

$$\sup_{(p,\nu)\in\mathfrak{D}^{high}_{rw}} \sup_{t\geq 0} \|U_\nu R^{high}_{rw}(t,p)U_{-\nu}\| < C \tag{4.44}$$

The appearance of the factor λ^2 is due to the fact that λ^2 multiplies $\mathcal{M}$ in the definition of the semigroup Λ_t.

Next, we derive estimates on Λ_t (see ee.g. the bound (4.55) below) starting from (4.41-4.42) and (4.43-4.44), without using explicitly the semigroup property of Λ_t. This is important since in the proof of Lemma 6.2, we will carry out an analogous derivation for objects which are not semigroups.

We choose $\kappa=(\kappa_L,\kappa_R)\in\mathbb{C}^d\times\mathbb{C}^d$ such that $\operatorname{Re}\kappa_L=\operatorname{Re}\kappa_R=0$ and we calculate, using relation (2.60),

$$\mathcal{J}_\kappa\Lambda_t\mathcal{J}_{-\kappa} = \int_{\mathbb{T}^d}^{\oplus} \mathrm{d}p\, U_\nu\, \{\Lambda_t\}_{p+\Delta p}\, U_{-\nu}, \qquad \text{with } \Delta p := \frac{\kappa_L-\kappa_R}{2} \quad \nu := \frac{\kappa_L+\kappa_R}{4} \tag{4.45}$$

where we are using the analyticity in (p,ν), see (4.35) and (4.43-4.44). Recall that $\{\Lambda\}_p$ acts on $\mathscr{G}^p\sim\mathscr{G}\sim L^2(\mathbb{T}^d,\mathscr{B}(\mathscr{S}))$. Our choice for κ ensures that Δp and ν are purely imaginary.

Next, we split the integration over $p\in\mathbb{T}^d$ into small fibers ($|p|<p^*_{rw}$) and large fibers ($|p|\geq p^*_{rw}$) by defining

$$I^{low} := \{p+\Delta p \,\big|\, p\in\mathbb{T}^d, |p|<p^*_{rw}\}, \qquad I^{high} := \{p+\Delta p\,\big|\, p\in\mathbb{T}^d, |p|\geq p^*_{rw}\} \tag{4.46}$$

Using the relations (4.41) and (4.42), we obtain

$$\mathcal{J}_\kappa \Lambda_t \mathcal{J}_{-\kappa} = \underbrace{\int_{I^{low}}^{\oplus} \mathrm{d}p\, \mathrm{e}^{\lambda^2 f_{rw}(p)t}\, U_\nu P_{rw}(p) U_{-\nu}}_{=:B_1} + \mathrm{e}^{-\lambda^2 g_{rw}^{low} t} \underbrace{\int_{I^{low}}^{\oplus} \mathrm{d}p\, U_\nu R_{rw}^{low}(p,t) U_{-\nu}}_{=:B_2} \tag{4.47}$$

$$+ \quad \mathrm{e}^{-\lambda^2 g_{rw}^{high} t} \underbrace{\int_{I^{high}}^{\oplus} \mathrm{d}p\, U_\nu R_{rw}^{high}(p,t) U_{-\nu}}_{=:B_3}$$

We establish decay properties of the operators $B_{1,2,3}$ in position representation. For B_2 and B_3 we proceed as follows. Recall the duality (2.61-2.62). By varying p and ν, and using the bounds (4.43-4.44), we obtain

$$\|(B_{2,3})_{x_L,x_R;x'_L,x'_R}\|_{\mathscr{S}} \leq C\mathrm{e}^{-\frac{\gamma}{2}\left|(x'_L+x'_R)-(x_L+x_R)\right|}\mathrm{e}^{-\gamma\left|(x_L-x_R)-(x'_L-x'_R)\right|} \tag{4.48}$$

for any $\gamma < \delta_{rw}$.

For B_1, we need a better bound, which is attained by exploiting the fact that $P_{rw}(p)$ is a rank-1 operator with a kernel of the form (recall the notation of (2.5))

$$P_{rw}(p)(k,k') = \Big|\left(\xi_{rw}(p)\right)(k)\Big\rangle\Big\langle(\tilde{\xi}_{rw}(p))(k')\Big|, \qquad \text{for some} \quad \xi_{rw}(p),\ \tilde{\xi}_{rw}(p) \in \mathscr{G} \tag{4.49}$$

where both $\xi_{rw}(p), \tilde{\xi}_{rw}(p)$ are bounded-analytic functions of k, k', respectively, in a strip of width δ_{rw}. This follows from boundedness and analyticity of $P_{rw}(p)$ by the same reasoning as in Lemma 3.4. By the definition of B_1 in (4.47) and (2.56), (2.60),

$$(B_1)_{x_L,x_R;x'_L,x'_R} = \int_{I^{low}} \mathrm{d}p\, \mathrm{e}^{\lambda^2 f_{rw}(p)t}\mathrm{e}^{-\mathrm{i}\frac{p}{2}((x'_L+x'_R)-(x_L+x_R))} \tag{4.50}$$

$$\int_{\mathbb{T}^d+\nu}\mathrm{d}k\int_{\mathbb{T}^d+\nu}\mathrm{d}k'\mathrm{e}^{-\mathrm{i}k(x_L-x_R)+\mathrm{i}k'(x'_L-x'_R)}P_{rw}(p)(k,k') \tag{4.51}$$

By the analyticity of $P_{rw}(p)(\cdot,\cdot)$ in both k and k', we derive, for $\gamma < \delta_{rw}$,

$$\|(B_1)_{x_L,x_R;x'_L,x'_R}\|_{\mathscr{S}} \leq C\mathrm{e}^{r_{rw}(\gamma,\lambda)t}\mathrm{e}^{-\frac{\gamma}{2}\left|(x'_L+x'_R)-(x_L+x_R)\right|}\mathrm{e}^{-\gamma|x_L-x_R|}\mathrm{e}^{-\gamma\left|x'_L-x'_R\right|} \tag{4.52}$$

where the function $r_{rw}(\gamma,\lambda)$ is given by

$$r_{rw}(\gamma,\lambda) := \lambda^2 \sup_{|\mathrm{Im}\, p|\leq\gamma, |\mathrm{Re}\, p|\leq p^*_{rw}} \mathrm{Re}\, f_{rw}(p). \tag{4.53}$$

Note that

$$r_{rw}(\gamma,\lambda) := O(\lambda^2)O(\gamma^2), \qquad \lambda\searrow 0, \gamma\searrow 0, \tag{4.54}$$

as follows from $\mathrm{Re}\, f_{rw}(p) \leq 0$. The bound (4.54) will be used to argue that the exponential blowup in time, given by $\mathrm{e}^{r_{rw}(\gamma,\lambda)t}$ can be compensated by the decay $\mathrm{e}^{-g_{rw}\lambda^2 t}$ by choosing γ small enough, see Section 5.

Putting the bounds on $B_{1,2,3}$ together, we arrive at

$$\begin{aligned}\|(\Lambda_t)_{x_L,x_R;x'_L,x'_R}\| \quad &\leq \quad C\mathrm{e}^{r_{rw}(\gamma,\lambda)t}\mathrm{e}^{-\frac{\gamma}{2}\left|(x'_L+x'_R)-(x_L+x_R)\right|}\mathrm{e}^{-\gamma|x_L-x_R|}\mathrm{e}^{-\gamma\left|x'_L-x'_R\right|}\\ &+ \quad C'\mathrm{e}^{-\lambda^2 g_{rw}t}\mathrm{e}^{-\frac{\gamma}{2}\left|(x'_L+x'_R)-(x_L+x_R)\right|}\mathrm{e}^{-\gamma\left|(x_L-x_R)-(x'_L-x'_R)\right|}\end{aligned} \tag{4.55}$$

for $\gamma < \delta_{rw}$ and with the rate $g_{rw} := \min\,(g_{rw}^{low}, g_{rw}^{high})$. The bound (4.55) is the main result of the present section and it will be used in a crucial way in the proofs. The importance of this bound is explained in Section 5.4.

For completeness, we note that a bound like

$$\|(\Lambda_t)_{x_L,x_R;x'_L,x'_R}\| \leq \mathrm{e}^{2\lambda^2 q_\varepsilon(\gamma)t}\mathrm{e}^{-\gamma|x'_L-x_L|}\mathrm{e}^{-\gamma|x'_R-x_R|} \tag{4.56}$$

can be derived simply from the fact that $\mathcal{J}_\kappa\mathcal{M}\mathcal{J}_\kappa$ is bounded for complex κ, see (4.6), since

$$\kappa_L \qquad \text{is conjugate to} \qquad (x'_L - x_L) \tag{4.57}$$

$$\kappa_R \qquad \text{is conjugate to} \qquad (x'_R - x_R), \tag{4.58}$$

5 Strategy of the proofs

In this section, we outline our strategy for proving the results in Section 3. We start by introducing and analyzing the space-time reservoir correlation function $\psi(x,t)$. Then we introduce a perturbation expansion for the reduced evolution $\mathcal{Z}_t$ (which involves the reservoir correlation function). Afterwards, we describe and motivate the temporal cutoff that we will put into the expansion. Finally, a plan of the proof is given.

5.1 Reservoir correlation function

A quantity that will play an important role in our analysis is the free reservoir correlation function $\psi(x,t)$, which we define next. Let

$$I^x_{\mathrm{SR}} := \int \mathrm{d}q\left(\phi(q)\mathrm{e}^{\mathrm{i}q\cdot x}1_x\otimes a_q + \overline{\phi(q)}\mathrm{e}^{-\mathrm{i}q\cdot x}1_x\otimes a^*_q\right) \tag{5.1}$$

where $1_x = 1_x(X)$ is the projection on $\mathscr{H}_{\mathrm{S}}$, acting as $(1_x\varphi)(x') = \delta_{x,x'}\varphi(x)$ for $\varphi \in l^2(\mathbb{Z}^d,\mathscr{S})$. The operator I^x_{SR} is the part of the system-reservoir coupling that acts at site x after setting the matrix $W \in \mathscr{B}(\mathscr{S})$ equal to 1 (recall that the matrix W describes the coupling of the internal degrees of freedom to the reservoir). We also define the time-evolved interaction term, with the time-evolution given by the free reservoir dynamics

$$I^x_{\mathrm{SR}}(t) := \mathrm{e}^{\mathrm{i}tH_{\mathrm{R}}}I^x_{\mathrm{SR}}\mathrm{e}^{-\mathrm{i}tH_{\mathrm{R}}} \tag{5.2}$$

The reservoir correlation function ψ is then defined as

$$\begin{aligned}\psi(x,t) &:= \rho^\beta_{\mathrm{R}}\left[I^x_{\mathrm{SR}}(t)I^0_{\mathrm{SR}}(0)\right],\\ &= \langle\phi^x, T_\beta\mathrm{e}^{\mathrm{i}t\omega}\phi\rangle_{\mathfrak{h}} + \langle\phi^x,(1+T_\beta)\mathrm{e}^{-\mathrm{i}t\omega}\phi\rangle_{\mathfrak{h}}\\ &= \int_{\mathbb{R}}\mathrm{d}\omega\hat{\psi}(\omega)\mathrm{e}^{\mathrm{i}\omega t}\int_{\mathbb{S}^{d-1}}\mathrm{d}s\,\mathrm{e}^{\mathrm{i}\omega s\cdot x}\end{aligned} \tag{5.3}$$

where $(\phi^x)(q) := \mathrm{e}^{\mathrm{i}q\cdot x}\phi(q)$ and $\mathbb{S}^{d-1}$ is the $d-1$-dimensional hypersphere of unit radius. The 'effective squared form factor' $\hat{\psi}$ was introduced in (2.27), and the density operator T_β in Section 2.3.3.

Assumptions 2.2 and 2.3 imply certain properties of the correlation function that will be primary ingredients of the proofs. We state these properties as lemmata. In fact, one could treat these properties as the very assumptions of our paper, since, in practice, Assumptions 2.2 and 2.3 will only be used to guarantee these properties, Lemmata 5.1 and 5.2. The straightforward proofs of Lemmata 5.1 and 5.2 are postponed to Appendix A.

The following lemma states that the free reservoir has exponential decay in t whenever $|x|/t$ is smaller than some speed v_*.

Lemma 5.1 (Exponential decay at 'subluminal' speed)**.** *Assume Assumptions 2.2 and 2.3. Then there are positive constants $v_* > 0, g_{\mathrm{R}} > 0$ such that*

$$|\psi(x,t)| \leq C\exp\left(-g_{\mathrm{R}}|t|\right), \quad \textit{if} \quad \frac{|x|}{t} \leq v_*, \qquad \textit{for some constant } C. \tag{5.4}$$

Property (5.4) is satisfied if the reservoir is 'relativistic', i.e., if the dispersion law $\omega(q)$ of the reservoir particles is linear in the momentum $|q|$, temperature β^{-1} is positive and the form factor ϕ satisfies the infrared regularity condition that $k \mapsto |\phi(k)|^2|k|$ is analytic in a strip around the real axis. The speed v^* has to be chosen strictly smaller than the propagation speed of the reservoir modes given by the slope of ω. In fact, the decay rate g_{R} vanishes when v^* approaches the propagation speed of the reservoir modes. Lemma 5.1 does **not** depend on the fact that the dimension $d \geq 4$.

Lemma 5.2 (Time-integrable correlations)**.** *Assume Assumptions 2.2 and 2.3. Then*

$$\int_{\mathbb{R}^+} \mathrm{d}t \sup_{x \in \mathbb{Z}^d} |\psi(x,t)| < \infty \tag{5.5}$$

This property is satisfied for non-relativistic reservoirs, with $\omega(q) \propto |q|^2$, in $d \geq 3$ and for relativistic reservoirs, with $\omega(q) \propto |q|$, in $d \geq 4$, provided that we choose the coupling to be sufficiently regular in the infrared.

□

5.2 The Dyson expansion

In this section, we set up a convenient notation to handle the Dyson expansion introduced in Lemma 2.5.

We define the group $\mathcal{U}_t$ on $\mathscr{B}(\mathscr{H}_{\mathrm{S}})$ by

$$\mathcal{U}_t S := \mathrm{e}^{-\mathrm{i}tH_{\mathrm{S}}} S \mathrm{e}^{\mathrm{i}tH_{\mathrm{S}}}, \qquad S \in \mathscr{B}(\mathscr{H}_{\mathrm{S}}), \tag{5.6}$$

and the operators $\mathcal{I}_{x,l}$, with $x \in \mathbb{Z}^d$ and $l \in \{L, R\}$ (L, R stand for '*left*' and '*right*'), as

$$\mathcal{I}_{x,l} S := \begin{cases} -\mathrm{i} \quad (1_x \otimes W)S & \text{if} \quad l = L \\ \ \ \mathrm{i} \quad S(1_x \otimes W) & \text{if} \quad l = R \end{cases} \qquad S \in \mathscr{B}(\mathscr{H}_{\mathrm{S}}). \tag{5.7}$$

where the operators $1_x \equiv 1_x(X)$ are projections on a lattice site $x \in \mathbb{Z}^d$, as used in Section 5.1.

We write (t_i, x_i, l_i), $i = 1, \ldots, 2n$ to denote $2n$ triples in $\mathbb{R} \times \mathbb{Z}^d \times \{L, R\}$ and we assume them to be ordered by the time coordinates, i.e., $t_i < t_{i+1}$. We evaluate the Lie-Schwinger series (2.43) using the properties (2.21-2.22-2.23), and we arrive at

$$\mathcal{Z}_t = \sum_{n \in \mathbb{Z}^+} \sum_{\pi \in \mathcal{P}_n} \int_{0<t_1<\ldots<t_{2n}<t} \mathrm{d}t_1 \ldots \mathrm{d}t_{2n} \zeta(\pi, (t_i, x_i, l_i)_{i=1}^{2n}) \mathcal{V}_t((t_i, x_i, l_i)_{i=1}^{2n}) \tag{5.8}$$

where $\pi \in \mathcal{P}_n$ are pairings, as in (2.23), and we define

$$\mathcal{V}_t((t_i, x_i, l_i)_{i=1}^{2n}) := \mathcal{U}_{t-t_{2n}} \mathcal{I}_{x_{2n}, l_{2n}} \ldots \mathcal{I}_{x_2, l_2} \mathcal{U}_{t_2 - t_1} \mathcal{I}_{x_1, l_1} \mathcal{U}_{t_1} \tag{5.9}$$

with $\mathcal{U}_t$ as in (5.6) and

$$\zeta(\pi, (t_i, x_i, l_i)_{i=1}^{2n}) := \prod_{(r,s) \in \pi} \lambda^2 \begin{cases} \psi(x_s - x_r, t_s - t_r) & l_r = l_s = L \\ \overline{\psi}(x_s - x_r, t_s - t_r) & l_r = l_s = R \\ \overline{\psi}(x_s - x_r, t_s - t_r) & l_r = L, l_s = R \\ \psi(x_s - x_r, t_s - t_r) & l_r = R, l_s = L \end{cases} \tag{5.10}$$

with the correlation function ψ as defined in (5.3). We recall the convention $r < s$ for each element of a pairing π. For $n = 0$, the integral in (5.8) is meant to be equal to $\mathcal{U}_t$. In Section 7, we will introduce some combinatorial concepts to deal with the pairings $\pi \in \mathcal{P}_n$ that are used in (5.8). For convenience, we will replace the variables $(\pi, (t_i, x_i, l_i)_{i=1}^{2n}) \in \mathcal{P}_n \times ([0,t] \times \mathbb{Z}^d \times \{L, R\})^{2n}$ by a single variable σ that carries the same information.

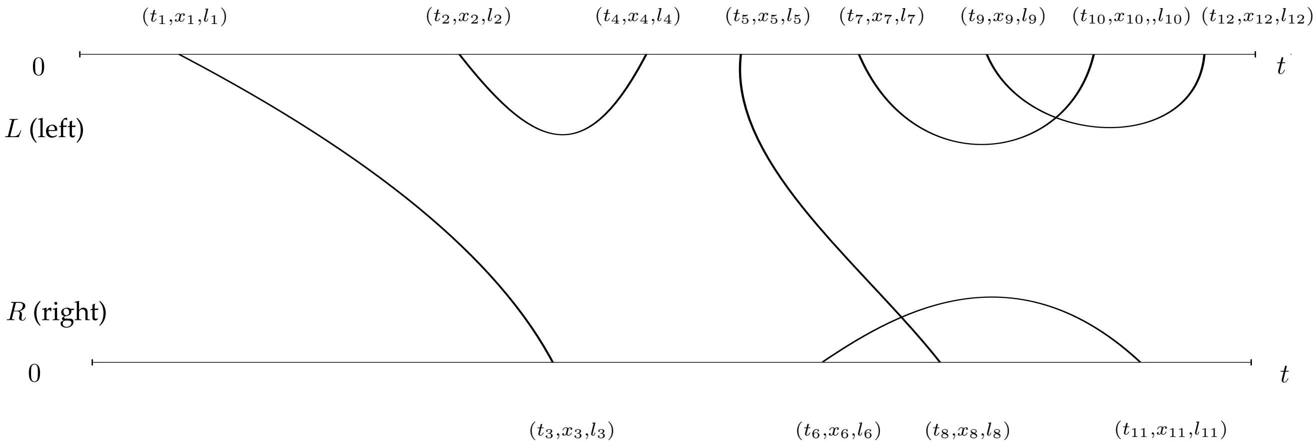


Figure 5: Graphical representation of a term contributing to the RHS of (5.8) with $\pi = \{(1,3),(2,4),(5,8),(6,10),(7,11),(9,12)\} \in \mathcal{P}_6$. The times t_i correspond to the position of the points on the horizontal axis.

Starting from this graphical representation, we can reconstruct the corresponding term in (5.8) - an operator on $\mathscr{B}_2(\mathscr{H}_{\mathrm{S}})$)- as follows

- To each straight line between the points (t_i, x_i, l_i) and (t_j, x_j, l_j), one associates the operators $\mathrm{e}^{\pm \mathrm{i}(t_j - t_i) H_{\mathrm{S}}}$, with $\pm$ being $-$ for $l_i = l_j = L$ and $+$ for $l_i = l_j = R$.
- To each point (t_i, x_i, l_i), one associates the operator $\lambda^2 \mathcal{I}_{x_i, l_i}$, defined in (5.7).
- To each curved line between the points (t_r, x_r, l_r) and (t_s, x_s, l_s), with $r < s$, we associate the factor $\psi^{\#}(x_s - x_r, t_s - t_r)$ with $\psi^{\#}$ being ψ or $\overline{\psi}$, depending on l_r, l_s, as prescribed in (5.10).

Rules like these are commonly called "Feynman rules" by physicists.

5.3 The cut-off model

In our model, the space-time correlation function $\psi(x,t)$ does not decay exponentially in time, uniformly in space, i.e,

$$\textit{there is no} \quad g > 0 \quad \textit{such that} \qquad \sup_{x \in \mathbb{Z}^d} |\psi(x,t)| \leq C \mathrm{e}^{-g|t|} \tag{5.11}$$

The impossibility of choosing the form factor ϕ or any other model parameter such that one has exponential decay is a fundamental consequence of local momentum conservation, as explained in Section 1.3.

However, if the correlation function $\psi(x,t)$ did decay exponentially, we could set up a perturbation expansion for $\mathcal{Z}_t$ around the Markovian limit Λ_t. Such a scheme was implemented in [32], building on an expansion introduced in [31].

In the present section, we modify our model by introducing a cutoff time τ into the correlation function $\psi(x,t)$. More concretely, we modify the perturbation expansion for $\mathcal{Z}_t$ by replacing

$$\psi(x,t), \qquad \longrightarrow \qquad 1_{|t| \leq \tau} \psi(x,t) \tag{5.12}$$

The cutoff time τ will be chosen as a function of λ satisfying

$$\tau(\lambda) \to \infty, \qquad \tau(\lambda)\lambda \to 0, \qquad \text{as } \lambda \searrow 0 \tag{5.13}$$

However, we will take care to keep τ explicit in the estimates, until Section 8 where the τ-dependence will often be hidden in generic constants $c(\gamma,\lambda), c'(\gamma,\lambda)$. With the cut-off in place, the correlation function $\psi(x,t)$ decays exponentially, uniformly in $x \in \mathbb{Z}^d$, i.e., obviously,

$$\sup_{x \in \mathbb{Z}^d} 1_{|t| \leq \tau} |\psi(x,t)| \leq C \mathrm{e}^{-\frac{|t|}{\tau}}. \tag{5.14}$$

The modified reduced dynamics obtained in this way will be called $\mathcal{Z}_t^\tau$.

That is;

$$\mathcal{Z}_t^\tau = \sum_{n\in\mathbb{Z}^+}\sum_{\pi\in\mathcal{P}_n}\int_{0<t_1<\ldots<t_{2n}<t} \mathrm{d}t_1\ldots\mathrm{d}t_{2n}\,\zeta_\tau(\pi,(t_i,x_i,l_i)_{i=1}^{2n})\,\mathcal{V}_t((t_i,x_i,l_i)_{i=1}^{2n}) \tag{5.15}$$

with

$$\zeta_\tau(\pi,(t_i,x_i,l_i)_{i=1}^{2n}) := \left(\prod_{(r,s)\in\pi} 1_{|t_s-t_r|\leq\tau}\right)\zeta(\pi,(t_i,x_i,l_i)_{i=1}^{2n}) \tag{5.16}$$

If τ is chosen to be independent of λ then one can analyze $\mathcal{Z}_t^\tau$ by the technique deployed in [32]. It turns out that for a λ-dependent τ, one can still analyze the cutoff model by the same techniques as long as $\lambda^2\tau(\lambda)\searrow 0$ as $\lambda\searrow 0$, which is satisfied by our choice (5.13). The analysis of $\mathcal{Z}_t^\tau$ is outlined in Lemma 6.1, in Section 6.1. The main conclusion of the treatment of the cutoff model is that

$$\textit{The cutoff reduced dynamics } \mathcal{Z}_t^\tau \textit{ is 'close' to the semigroup } \Lambda_t \tag{5.17}$$

This conclusion is partially embodied in Lemma 6.2. For the sake of this explanatory chapter, one can identify $\mathcal{Z}_t^\tau$ with Λ_t.

The reason why it is useful to treat the cutoff model first, is that we will perform a renormalization step, effectively replacing the free evolution $\mathcal{U}_t$ in the expansion (5.8) by the cutoff reduced dynamics $\mathcal{Z}_t^\tau$. The benefit of such a replacement is explained in Section 5.4.

5.4 Exponential decay for the renormalized correlation function

5.4.1 The joint system-reservoir correlation function

We recall that the free reservoir correlation function $\psi(x,t)$ does not decay exponentially in t, uniformly in x. This was mentioned already in Section 5.3 and it motivated the introduction of the temporal cutoff τ.

In the perturbation expansion for the reduced evolution $\mathcal{Z}_t$, the correlation function $\psi(x,t)$ models the propagation of reservoir modes over a space-time 'distance' (x,t) and it occurs together with terms describing the propagation of the particle. Let us look at the lowest-order terms in the expansion of $\mathcal{Z}_t$, introduced in Section 5.2 above;

$$\mathcal{Z}_t = \lambda^2\int_{0<t_1<t_2<t}\mathrm{d}t_2\mathrm{d}t_1\sum_{x_1,x_2,l_1,l_2}\psi^{\#}(x_2-x_1,t_2-t_1)\quad \mathcal{U}_{t-t_2}\mathcal{I}_{x_2,l_2}\mathcal{U}_{t_2-t_1}\mathcal{I}_{x_1,l_1}\mathcal{U}_{t_1} + \text{higher orders in } \lambda \tag{5.18}$$

with $\psi^{\#}$ being ψ or $\overline{\psi}$, as prescribed by the rules in (5.10). It is natural to ask whether the 'joint correlation function'

$$\psi^{\#}(x_2-x_1,t_2-t_1)\quad \mathcal{I}_{x_2,l_2}\mathcal{U}_{t_2-t_1}\mathcal{I}_{x_1,l_1} \tag{5.19}$$

has better decay properties than $\psi(x,t)$ by itself. In particular, we ask whether (5.19) is exponentially decaying in t_2-t_1, uniformly in x_2-x_1. This turns out to be the case only if $l_1=l_2$ since in that case, the question essentially amounts to bounding

$$|\psi(x_2-x_1,t_2-t_1)|\times\left\|\left(\mathrm{e}^{\pm\mathrm{i}(t_2-t_1)\lambda^2\varepsilon(P)}\right)(x_1,x_2)\right\| \tag{5.20}$$

The expression (5.20) has exponential decay in time because

- For speed $\left|\frac{x_2-x_1}{t_2-t_1}\right|$ greater than some $v>0$, we estimate

$$\left\|\left(\mathrm{e}^{\pm\mathrm{i}(t_2-t_1)\lambda^2\varepsilon(P)}\right)(x_1,x_2)\right\|\leq \mathrm{e}^{-(\gamma v-\lambda^2 q_\varepsilon(\gamma))|t_2-t_1|},\qquad \text{for } 0<\gamma\leq\delta_\varepsilon \tag{5.21}$$

  with $\delta_\varepsilon, q_\varepsilon(\cdot)$ as in (2.12) and Assumption 2.1. Hence, for fixed v, one can choose γ so as to make the exponent on the RHS of (5.21) negative, for λ small enough.

- For speeds $\left|\frac{x_2-x_1}{t_2-t_1}\right|$ smaller than $v^*>0$, the reservoir correlation function $\psi(x_2-x_1,t_2-t_1)$ decays with rate g_{R}, as asserted in Lemma 5.1 with v^* as defined therein.

When $l_1 \neq l_2$ in (5.19), there is no decay at all from $\mathcal{U}_{t_2-t_1}$, in other words, the 'matrix element'

$$(\mathcal{U}_{t_2-t_1})_{x_L,x_R;x'_L,x'_R} \tag{5.22}$$

is obviously not decaying in the variables $x_L-x'_R$ or $x_R-x'_L$, since it is a function of $x_L-x'_L$ and $x_R-x'_R$ only. Hence, for $l_2 \neq l_1$, the joint correlation function (5.19) has as poor decay properties as the reservoir correlation function $\psi(x,t)$.

The situation is summarized in the following table

joint $\mathrm{S}-\mathrm{R}$ correlation fct.		
$\lvert x\rvert/t>v^*$		$\lvert x\rvert/t\leq v^*$
$l_1\neq l_2$	$l_1=l_2$	l_1,l_2 arbitrary
No exp. decay	exp. decay from $\lVert \mathrm{e}^{\pm \mathrm{i}tH_{\mathrm{S}}}(0,x)\rVert$	exp. decay from $\psi(x,t)$

5.4.2 Renormalized joint correlation function

The bad decay property of the joint correlation function (5.19) suggests to perform a renormalization step, replacing the free propagator $\mathcal{U}_t$ by the cutoff reduced dynamics $\mathcal{Z}_t^{\tau}$, for which (5.19) has exponential decay when $l_1\neq l_2$. The cutoff reduced dynamics $\mathcal{Z}_t^{\tau}$ was introduced in Section 5.3, where we argued that it is well approximated by the Markov semigroup Λ_t. Hence, we replace the group $\mathcal{U}_t$ by the semigroup Λ_t in (5.19), thus obtaining a 'renormalized joint system-reservoir correlation function'. We then check that the so-defined renormalized correlation function has exponential decay in time, uniformly in space: For λ small enough,

$$\left|\psi(x'_{l_2}-x_{l_1},t_2-t_1)\right| \times \left\|(\Lambda_{t_2-t_1})_{x_L,x_R;x'_L,x'_R}\right\| \leq \mathrm{e}^{-t\lambda^2 g_{rw}}, \qquad \text{for} \qquad l_1,l_2\in\{L,R\} \tag{5.23}$$

with the decay rate g_{rw} as in (4.55). To verify (5.23), we assume for concreteness that $l_1=L$ and $l_2=R$, and we estimate by the triangle inequality

$$|x'_R-x_L| \leq \frac{1}{2}|x'_R-x'_L| + \frac{1}{2}|(x'_R+x'_L)-(x_R+x_L)| + \frac{1}{2}|x_R-x_L| \tag{5.24}$$

We note that the three terms on the RHS of (5.24) correspond (up to factors $\frac{1}{2}$) to the three spatial arguments multiplying γ in the first line of (4.55). By (5.24), at least one of these terms is larger than $\frac{1}{3}|x'_R-x_L|$. Hence we dominate (4.55) by replacing that particular term by $\frac{1}{3}|x'_R-x_L|$. Setting all other spatial arguments in (4.55) equal to zero, we obtain

$$\|(\Lambda_t)_{x_L,x_R;x'_L,x'_R}\| \leq C\mathrm{e}^{r_{rw}(\gamma,\lambda)t}\mathrm{e}^{-\frac{\gamma}{6}|x'_R-x_L|} + C'\mathrm{e}^{-(\lambda^2 g_{rw})t} \tag{5.25}$$

Assuming that $|x'_R-x_L|\geq v^*|t_2-t_1|$ and using that $r_{rw}(\gamma,\lambda)=O(\gamma^2)O(\lambda^2)$, see (4.54), we choose γ such that the first term of (5.25) decays exponentially in t_2-t_1 with a rate or order 1. Hence, at high speed ($\geq v^*$) (5.23) is satisfied. At low speed ($\leq v^*$), (5.23) holds by the exponential decay of ψ and the bound $\|\Lambda_t\|\leq C\mathrm{e}^{O(\lambda^2)t}$, which is easily derived from (4.55).

For $l_1=l_2$, we can apply the same reasoning, and hence (5.23) is proven in general. However, in the case $l_1=l_2$, the proof is actually simpler. We can follow the same strategy as used for bounding (5.20), but replacing the propagation estimate (2.12) for $\mathcal{U}_t$ by the analogous estimate (4.6) for Λ_t. Indeed, the exponential decay in the case $l_1=l_2$ was already present without the coupling to the reservoir, as explained in Section 5.4.1, whereas the decay in the case $l_1\neq l_2$ is a nontrivial consequence of the decoherence induced by the reservoir.

renormalized $\mathrm{S}-\mathrm{R}$ correlation fct.		
$\lvert x\rvert/t>v^*$		$\lvert x\rvert/t\leq v^*$
$l_1\neq l_2$	$l_1=l_2$	l_1,l_2 arbitrary
exp. decay from decoherence of Λ_t	exp. decay from $\lVert \mathrm{e}^{\pm \mathrm{i}tH_{\mathrm{S}}}(0,x)\rVert$	exp. decay from $\psi(x,t)$

Along the same line, we note that the decay rate in (5.23) cannot be made greater than $O(\lambda^2)$, since the effect of the reservoir manifests itself only after a time $O(\lambda^{-2})$. This should be contrasted with the decay rate for (5.20), which can be chosen to be independent of λ.

5.5 The renormalized model

We have argued in the previous section that it makes sense to evaluate the perturbation expansion (5.8) in two steps by introducing a cutoff τ for the temporal arguments of the correlation function ζ. The resulting cutoff reduced evolution $\mathcal{Z}_t^\tau$ was described in Section 5.3. By reordering the perturbation expansion, we are able to rewrite the reduced evolution $\mathcal{Z}_t$ approximatively as

$$\mathcal{Z}_t \approx \lambda^2 \int_{\substack{0<t_1<t_2<t \\ |t_2-t_1|>\tau}} \mathrm{d}t_2 \mathrm{d}t_1 \sum_{x_1,x_2,l_1,l_2} \psi^{\#}(x_2-x_1, t_2-t_1) \quad \mathcal{Z}^\tau_{t-t_2} \mathcal{I}_{x_2,l_2} \mathcal{Z}^\tau_{t_2-t_1} \mathcal{I}_{x_1,l_1} \mathcal{Z}^\tau_{t_1} + \text{higher orders in } \lambda \tag{5.26}$$

where the restriction that $t_2 - t_1 > \tau$ reflects the fact that the short diagrams have been resummed. Note that it is somewhat misleading to call the remainder of the perturbation series 'higher order in λ', since τ will be λ-dependent, too.

The main tools in dealing with the renormalized model are

1) The exponential decay of the renormalized joint correlation function, as outlined in Section 5.4. This property holds true thanks to the decoherence in the Markov semigroup Λ_t and the exponential decay for low ('subluminal') speed of the bare reservoir correlation function. The latter is a consequence of the fact that the dispersion law of the reservoir modes is linear (see Lemma 5.1). The necessity of the exponential decay of the renormalized joint correlation function for the final analysis will become apparent in Lemma 9.4.
2) The integrability in time of the correlation function, uniformly in space, as stated in Lemma 5.2. This property allows us to sum up all subleading diagrams in the renormalized model. This will be made more explicit in Section 9.2, in particular in Lemma 9.3.

The most convenient description of the renormalized model will be reached at the end of Section 8 and the beginning of Section 9, where a representation in the spirit of (5.26) is discussed. The treatment of the renormalized model is contained in Section 9.

5.6 Plan of the proofs

In Section 6, we present the analysis of the cutoff reduced dynamics $\mathcal{Z}_t^\tau$ and the full reduced dynamics $\mathcal{Z}_t$, starting from bounds that are obtained in later sections. The main ingredient of this analysis is spectral perturbation theory, contained in Appendix B.

In Section 7, we introduce Feynman diagrams and we use them to derive convenient expressions for the cutoff reduced dynamics $\mathcal{Z}_t^\tau$ and the full reduced dynamics $\mathcal{Z}_t$. We will distinguish between *long* and *short* diagrams. The cutoff reduced dynamics contains only short diagrams.

Section 8 contains the analysis of the short diagrams. In particular, we prove the bounds on $\mathcal{Z}_t^\tau$, which were used in Section 6.

In Section 9, we deal with the long diagrams. In particular, we prove the bounds on $\mathcal{Z}_t$ from Section 6. At the end of the paper, in Section 9.4, we collect the most important constants and parameters of our analysis. A flow chart of the proofs is presented in Figure 6.

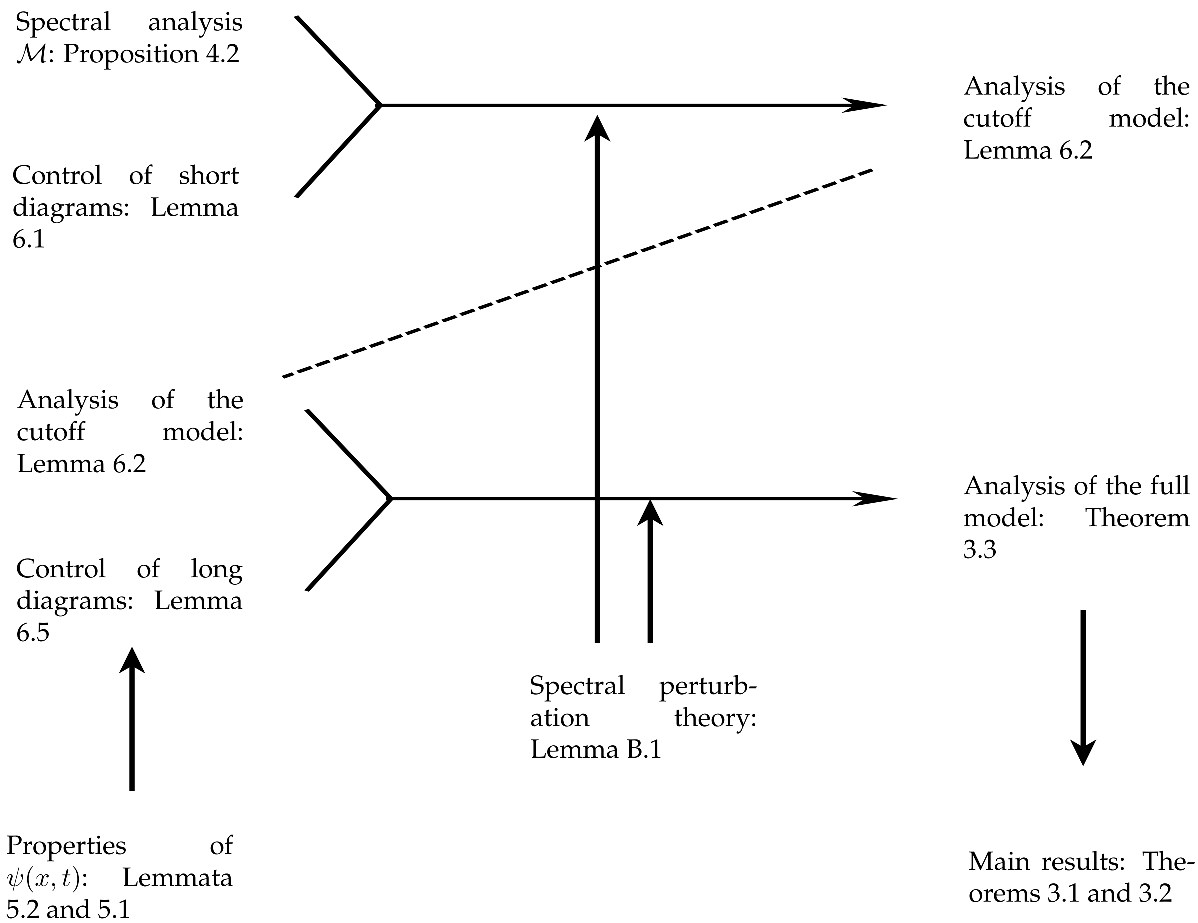


Figure 6: A flow chart of the proofs. The arrows mean "implies". The arrows pointing to arrows specify the proof of the implication.

6 Large time analysis of the reduced evolution $\mathcal{Z}_t$ and the cutoff reduced evolution $\mathcal{Z}_t^\tau$

In this section, we analyze the evolution operators $\mathcal{Z}_t$ and $\mathcal{Z}_t^\tau$ starting from bounds on their Laplace transforms

$$\mathcal{R}^\tau(z) := \int\limits_{\mathbb{R}^+} \mathrm{d}t\, \mathrm{e}^{-tz} \mathcal{Z}_t^\tau \tag{6.1}$$

and

$$\mathcal{R}(z) := \int\limits_{\mathbb{R}^+} \mathrm{d}t\, \mathrm{e}^{-tz} \mathcal{Z}_t. \tag{6.2}$$

These bounds are proven by diagrammatic expansions in Sections 7, 8 and 9. However, the present section is written in such a way that one can ignore these diagrammatic expansions and consider the bounds on $\mathcal{R}^\tau(z)$ and $\mathcal{R}(z)$ as an abstract starting point. Our results, Lemma 6.2, and Theorem 3.3, follow from these bounds by an application of the inverse Laplace transform and spectral perturbation theory. For convenience, these tools are collected in Lemma B.1 in Appendix B.

6.1 Analysis of $\mathcal{Z}_t^\tau$

Our main tool in the study of $\mathcal{R}^\tau(z)$ is Lemma 6.1 below. Loosely speaking, the important consequence of this lemma is the fact that we can represent the Laplace transform $\mathcal{R}^\tau$, defined in (6.1), as

$$\mathcal{R}^\tau(z) = (z - (-\mathrm{iad}(Y) + \lambda^2 \mathcal{M} + \mathcal{A}^\tau(z)))^{-1}, \tag{6.3}$$

where the operator $\mathcal{A}^\tau(z)$ is "small" wrt. $\lambda^2\mathcal{M}$, in a sense specified by the theorem. Note that if we set $\mathcal{A}^\tau(z) = 0$, then the RHS of (6.3) is the Laplace transform of the Markov semigroup Λ_t. This is consistent with the claim that $\mathcal{Z}_t^\tau$ is 'close to' Λ_t.

The subscripts 'ld' and 'ex', introduced below, stand for "ladder" and "excitations", respectively. These subscripts will acquire an intuitive meaning in Section 7 when the diagrammatic representation of the expansion is introduced. The (sub)superscript τ indicates the dependence on the cutoff τ, but sometimes we will also use the (sub)superscript c. This will be done for quantities that are designed for the cut-off model but that do not necessarily change when τ is varied. Lemma 6.1 can be stated for any τ, but, as announced, it will be used for a λ-dependent τ.

Lemma 6.1. *For λ small enough, there are operators $\mathcal{R}^\tau_{\mathrm{ex}}(z)$ and $\mathcal{R}^\tau_{\mathrm{ld}}(z)$ in $\mathscr{B}(\mathscr{B}_2(\mathscr{H}_{\mathrm{S}}))$, depending on λ and τ, satisfying the following properties:*

1) *For $\mathrm{Re}\, z$ sufficiently large, the integral in* (6.1) *converges absolutely in $\mathscr{B}(\mathscr{B}_2(\mathscr{H}_{\mathrm{S}}))$ and*

$$\mathcal{R}^\tau(z) = (z - (-\mathrm{iad}(H_{\mathrm{S}}) + \mathcal{R}^\tau_{\mathrm{ld}}(z) + \mathcal{R}^\tau_{\mathrm{ex}}(z)))^{-1}. \tag{6.4}$$

2) *The operators $\mathcal{R}^\tau_{\mathrm{ld}}(z), \mathcal{R}^\tau_{\mathrm{ex}}(z)$ are analytic in z in the domain $\mathrm{Re}\, z > -\frac{1}{2\tau}$. Moreover, there is a positive constant $\delta_1 > 0$ such that*

$$\sup_{|\mathrm{Im}\,\kappa_{L,R}| \leq \delta_1, \mathrm{Re}\, z > -\frac{1}{2\tau}} \left\{ \begin{array}{lcll} \|\mathcal{J}_\kappa \mathcal{R}^\tau_{\mathrm{ex}}(z) \mathcal{J}_{-\kappa}\| & = & O(\lambda^2)O(\lambda^2\tau), & \lambda^2\tau \searrow 0, \lambda \searrow 0 \\ \|\mathcal{J}_\kappa \mathcal{R}^\tau_{\mathrm{ld}}(z) \mathcal{J}_{-\kappa}\| & \leq & \lambda^2 C & \end{array} \right. \tag{6.5}$$

3) *Recall the operator $\mathcal{L}(z)$, introduced in Section 4.1. It satisfies*

$$\sup_{|\mathrm{Im}\,\kappa_{L,R}| \leq \delta_1, \mathrm{Re}\, z \geq 0} \left\| \mathcal{J}_\kappa \left(\mathcal{R}^\tau_{\mathrm{ld}}(z) - \lambda^2 \mathcal{L}(z) \right) \mathcal{J}_{-\kappa} \right\| \leq \lambda^2 C \int_\tau^{+\infty} \mathrm{d}t \sup_x |\psi(x,t)| + \lambda^4 \tau C' \tag{6.6}$$

The proof of this lemma is given in Section 8.

From Lemma 6.1, one can deduce, by spectral methods, that $\mathcal{Z}_t^\tau$ inherits some of the properties of the Markovian dynamics Λ_t. Instead of stating explicitly all possible results about $\mathcal{Z}_t^\tau$, we restrict our attention to Lemma 6.2, in particular, to the bound (6.7). This bound is the analogue of the bound (4.55) for the semigroup dynamics Λ_t, and it will be used heavily in the analysis of $\mathcal{Z}_t$ in Section 8.

Lemma 6.2. *Let the cutoff reduced evolution $\mathcal{Z}_t^\tau$ be as defined in Section 5.3, with the cutoff time $\tau = \tau(\lambda)$ satisfying* (5.13). *Then there are positive numbers $\delta_c > 0$, $\lambda_c > 0$ and $g_c > 0$ such that, for $0 < |\lambda| < \lambda_c$ and $\gamma \leq \delta_c$,*

$$\begin{array}{rcl} \left\| (\mathcal{Z}_t^\tau)_{x_L,x_R;x'_L,x'_R} \right\|_{\mathscr{B}_2(\mathscr{S})} & \leq & c^1_{\mathcal{Z}}\, \mathrm{e}^{r_\tau(\gamma,\lambda)t} \mathrm{e}^{-\frac{\gamma}{2}\left|(x'_L+x'_R)-(x_L+x_R)\right|} \mathrm{e}^{-\gamma|x_L-x_R|} \mathrm{e}^{-\gamma\left|x'_L-x'_R\right|} \\ & + & c^2_{\mathcal{Z}}\, \mathrm{e}^{-\lambda^2 g_c t} \mathrm{e}^{-\frac{\gamma}{2}\left|(x'_L+x'_R)-(x_L+x_R)\right|} \mathrm{e}^{-\gamma\left|(x_L-x_R)-(x'_L-x'_R)\right|}, \end{array} \tag{6.7}$$

for constants $c^1_{\mathcal{Z}}, c^2_{\mathcal{Z}} > 0$, and with

$$r_\tau(\gamma,\lambda) = O(\lambda^2)O(\gamma^2) + o(\lambda^2) \qquad \lambda \searrow 0, \gamma \searrow 0 \tag{6.8}$$

where the bound $o(\lambda^2)$ is uniform for $\gamma \leq \delta_c$.

The constants $\delta_c > 0$ and decay rate $g_c > 0$ are in general smaller than the analogues δ_{rw} and g_{rw} in the bound (4.55).

Proof. We apply Lemma B.1 in Appendix B with $\epsilon := \lambda^2$ and

$$V(t,\epsilon) \quad := \quad U_\nu \left\{\mathcal{Z}^\tau_t\right\}_p U_{-\nu} \tag{6.9}$$

$$A_1(z,\epsilon) \quad := \quad U_\nu \Big(\left\{\mathcal{R}^\tau_{ex}(z)\right\}_p + \left\{\mathcal{R}^\tau_{ld}(z)\right\}_p - \lambda^2 \mathrm{i} \left\{\mathrm{ad}(\varepsilon(P))\right\}_p \Big) U_{-\nu} \tag{6.10}$$

$$N \quad := \quad U_\nu \left\{\mathcal{M}\right\}_p U_{-\nu} \tag{6.11}$$

$$B \quad := \quad -U_\nu \left\{\mathrm{ad}(Y)\right\}_p U_{-\nu} \simeq -\mathrm{ad}(Y) \tag{6.12}$$

and $(p,\nu) \in \mathfrak{D}^{low}_c$ with

$$\mathfrak{D}^{low}_c := \mathfrak{D}^{low}_{rw} \cap \left\{ |\mathrm{Im}\, p| \leq \min(\delta_1,\delta_\varepsilon), \mathrm{Im}\, \nu \leq \frac{1}{2}\min(\delta_1,\delta_\varepsilon) \right\} \tag{6.13}$$

The set $\mathfrak{D}^{low}_{rw}$ has been defined before Proposition 4.2, the bound on p,ν involving δ_1 ensures that we can convert the domain of analyticity in the variable κ in Lemma 6.1 into a domain of analyticity in the variables (p,ν), via the relation (2.60). Similarly, the bound on p,ν involving δ_ε ensures that

$$\sup_{(p,\nu)\in\mathfrak{D}^{low}_c} \left\| U_\nu \left\{\mathrm{ad}(\varepsilon(P))\right\}_p U_{-\nu} \right\| \leq C \tag{6.14}$$

as a consequence of the bound on $\mathcal{J}_\kappa \mathrm{ad}(\varepsilon(P))\mathcal{J}_{-\kappa}$ provided by Assumption 2.1 and eq. (2.12).

We now check, step by step, the conditions of Lemma B.1. First, the continuity of $V(t,\epsilon)$ and the bound (B.1) follow from Lemma 2.5 and Statement 1) of Lemma 6.1. The relation (B.4) is Statement 1) of Lemma 6.1
Condition 1) of Lemma B.1 is trivially satisfied since Y is a Hermitian matrix on a finite-dimensional space.

To check **Condition 2)** of Lemma B.1, we choose g_A as $g_A = 2g^{low}_{rw}$ and we will actually show that the bound (B.6), which is required in the region $\mathrm{Re}\, z > -\lambda^2 g_A$, holds in the region $\mathrm{Re}\, z > -1/(2\tau)$, as long as $\lambda^2\tau$ is small enough. By Cauchy's formula, this implies that

$$\frac{\partial}{\partial z} A_1(z,\lambda) = \frac{1}{2\pi i} \oint_{\mathcal{C}_z} \mathrm{d}z' \frac{A_1(z',\lambda)}{(z-z')^2} = O(\lambda^2\tau), \qquad \text{for}\, \mathrm{Re}\, z > -\lambda^2 g_A, \tag{6.15}$$

with $\mathcal{C}_z$ a circle of radius $O(1/\tau)$ centered at z. Hence (B.7) follows.

To check (B.6), we use the bound (6.14) for $\mathrm{ad}(\varepsilon(P))$. The boundedness of the other terms in $A_1(z,\lambda)$ follows immediately from (6.5) .
Condition 3) contains conditions on the spectrum of $\mathcal{M}_p$ that are satisfied thanks to Proposition 4.2. It remains to check (B.8). By the bound on $\mathcal{R}^\tau_{ex}(z)$ in (6.5), it suffices to check that, for any $a \in \mathrm{sp}(\mathrm{ad}(Y))$;

$$1_a(\mathrm{ad}(Y))\mathcal{J}_\kappa \Big(\lambda^2\mathcal{M} - (-\lambda^2 \mathrm{i}\mathrm{ad}(\varepsilon(P)) + \mathcal{R}^\tau_{ld}(-\mathrm{i}a)) \Big) \mathcal{J}_{-\kappa} 1_a(\mathrm{ad}(Y)) = o(\lambda^2), \qquad \text{as}\lambda \to 0 \tag{6.16}$$

This follows by the estimate in (6.6) and the relation between $\mathcal{M}$ and $\mathcal{L}$ in (4.4). Note that we used that $\tau(\lambda) \to \infty$ as $\lambda \to 0$ to get $o(\lambda^2)$ from the estimate (6.6).

Hence, we can apply Lemma B.1 and we obtain a number $f_\tau(p,\lambda)$, a rank-one projector $P_\tau(p,\lambda)$ and a family of operators $R^{low}_\tau(t,p,\lambda)$ such that

$$\left\{\mathcal{Z}^\tau_t\right\}_p = \mathrm{e}^{f_\tau(p,\lambda)t} P_\tau(p,\lambda) + \mathrm{e}^{-(\lambda^2 g^{low}_c)t} R^{low}_\tau(t,p,\lambda) \tag{6.17}$$

for some $g^{low}_c > 0$ (which can be chosen arbitrarily close to g^{low}_{rw} by taking $|\lambda|$ small enough), and such that

$$\sup_{(p,\nu)\in\mathfrak{D}^{low}_c} \| U_\nu P_\tau(p,\lambda) U_{-\nu} \| \leq C \tag{6.18}$$

$$\sup_{(p,\nu)\in\mathfrak{D}^{low}_c} \sup_{t\geq 0} \| U_\nu R^{low}_\tau(p,t) U_{-\nu} \| \leq C \tag{6.19}$$

$$\mathrm{Re}\, f_\tau(p,\lambda) > -\lambda^2 g^{low}_c \tag{6.20}$$

The above reasoning applies to small fibers, since we use the spectral analysis of Proposition 4.2. We now establish a simpler result about the cut-off reduced evolution $\{\mathcal{Z}_t^{\tau}\}_p$, for large fibers. Let

$$\mathfrak{D}_c^{high} := \mathfrak{D}_{rw}^{high} \cap \left\{ |\mathrm{Im}\, p| \leq \min(\delta_1, \delta_\varepsilon), \mathrm{Im}\, \nu \leq \frac{1}{2}\min(\delta_1, \delta_\varepsilon) \right\} \tag{6.21}$$

Although for $(p,\nu) \in \mathfrak{D}_c^{high}$, we cannot apply Lemma B.1, we can still apply Lemma B.2 to conclude that, for λ small enough, the singularities of $\{\mathcal{R}^{\tau}(z)\}_p$ in the domain, say, $\mathrm{Re}\, z > -2\lambda^2 g_{rw}^{high}$ lie at a distance $o(\lambda^2)$ from $\mathrm{sp}\mathcal{M}_p$. One can then easily prove that $\{\mathcal{R}^{\tau}(z)\}_p$ is bounded-analytic in a domain of the form $\mathrm{Re}\, z > -\lambda^2 g_{rw}^{high} + o(\lambda^2)$ and hence

$$\{\mathcal{Z}_t^{\tau}\}_p = R_{\tau}^{high}(p,t)\mathrm{e}^{-(\lambda^2 g_c^{high})t} \tag{6.22}$$

with a rate $g_c^{high} > 0$ (which can be chosen arbitrarily close to g_{rw}^{high} by making λ small enough) and

$$\sup_{(p,\nu)\in\mathfrak{D}_c^{high}} \sup_{t\geq 0} \|U_\nu R_{\tau}^{high}(t,p,\lambda)U_{-\nu}\| \leq C \tag{6.23}$$

Finally, we note that one can easily find a constant δ_c such that $\mathfrak{D}_c^{low}$ and $\mathfrak{D}_c^{high}$ are of the form (4.31) and (4.32) with the parameters δ_c instead of δ_{rw} (the parameter p_{rw}^* does not need to be readjusted).

With the information on $\mathcal{Z}_t^{\tau}$ obtained above, we are able to prove the bound (6.7) by the same reasoning as we employed in the lines following Proposition 4.2 to derive the bound (4.55).

The function $r_\tau(\gamma,\lambda)$ in the statement of Lemma 6.2 is determined as

$$r_\tau(\gamma,\lambda) := \sup_{p\in\mathbb{T}^d, |p|\leq\gamma} \mathrm{Re}\, f_\tau(p,\lambda) \tag{6.24}$$

and the bound (6.8) follows by (4.54) and

$$f_\tau(p,\lambda) - \lambda^2 f(p) = o(\lambda^2), \qquad \lambda \searrow 0, \tag{6.25}$$

which follows from (B.13) in Lemma B.1. The decay rate g_c is chosen as$g_c := \min(g_c^{low}, g_c^{high})$. This concludes the proof of Lemma 6.2.

□

We close this section with two remarks which are however not necessary for an understanding of the further stages of the proofs.

Remark 6.3. *As apparent from the bound* (6.6)*, one cannot take $\tau \equiv const$, since in that case, this bound becomes $O(\lambda^2)$ instead of $o(\lambda^2)$. This would mean that there is a difference of $O(\lambda^2)$ between $\mathcal{Z}_t^{\tau}$ and Λ_t, whereas the important terms in Λ_t are themselves of $O(\lambda^2)$. This is however not an essential point: as one can see from the classification of diagrams in Section 7, one could easily modify the definition of the cutoff model such that $\mathcal{Z}_t^{\tau}$ is close to Λ_t even at $\tau \equiv const$. This can be achieved by performing the cutoff on the* non-ladder *diagrams only, which is a notion introduced in Section 7. The true reason why τ must diverge to ∞ when $\lambda \searrow 0$ will become clear in the proof of Lemma 6.5, in Section 9.3.*

Remark 6.4. *One is tempted to say that any claim that is made about $\mathcal{Z}_t$ in Section 3 could be stated for $\mathcal{Z}_t^{\tau}$ as well. While this is correct for Theorem 3.3, it fails for Proposition 3.5. The reason is that the identity $f(p=0,\lambda)=0$ follows from the fact that $\mathcal{Z}_t$ conserves the trace of density matrices, as it is the reduced dynamics of a unitary evolution. This is not true for $\mathcal{Z}_t^{\tau}$, and hence we cannot prove (or even expect) that $f_\tau(p=0,\lambda)=0$.*

6.2 Spectral analysis of $\mathcal{Z}_t$

In this section, we state Lemma 6.5, the $\tau = \infty$ analogue of Lemma 6.1. This Lemma leads to our main result, Theorem 3.3, via reasoning that is almost identical to the one that led from Lemma 6.1 to Lemma 6.2.

Essentially (and analogously to Lemma 6.1), Lemma 6.5 states that the Laplace transform $\mathcal{R}(z)$, defined in (6.2), is of the form

$$\mathcal{R}(z) = (z - (-\mathrm{i}\mathrm{ad}(Y) + \lambda^2\mathcal{M} + \mathcal{A}(z)))^{-1} \tag{6.26}$$

where $\mathcal{A}(z)$ is 'small' w.r.t. $\lambda^2\mathcal{M}$.

Lemma 6.5. *There is an operator $\mathcal{R}_{\mathrm{ex}}(z) \in \mathscr{B}(\mathscr{B}_2(\mathscr{H}_{\mathrm{S}}))$, depending on λ and satisfying the following properties, for λ small enough:*

1) *For $\mathrm{Re}\, z$ sufficiently large, the integral in* (6.2) *converges absolutely in $\mathscr{B}(\mathscr{B}_2(\mathscr{H}_{\mathrm{S}}))$ and*
$$\mathcal{R}(z) = (\mathcal{R}^{\tau}(z)^{-1} - \mathcal{R}_{\mathrm{ex}}(z))^{-1} \tag{6.27}$$
where $\mathcal{R}^{\tau}(z)$ was introduced in (6.1) *and $\tau = \tau(\lambda)$ was defined in* (5.13).
2) *There are positive constants δ_{ex}, g_{ex} such that the operator $\mathcal{R}_{ex}(z)$ is analytic in z in the domain $\mathrm{Re}\, z > -\lambda^2 g_{ex}$ and*
$$\sup_{|\mathrm{Im}\, \kappa_{L,R}| \leq \delta_{ex}, \mathrm{Re}\, z > -\lambda^2 g_{ex}} \|\mathcal{J}_{\kappa} \mathcal{R}_{\mathrm{ex}}(z) \mathcal{J}_{-\kappa}\| = o(\lambda^2), \qquad \lambda \searrow 0. \tag{6.28}$$

The proof of Lemma 6.5 is contained in Section 8. Starting from Lemma 6.5, we can prove our main result, Theorem 3.3, by the spectral analysis outlined in Appendix B.

Proof of Theorem 3.3 We apply Lemma B.1 with $\epsilon := \lambda^2$ and
$$V(t,\epsilon) \;:=\; U_{\nu} \left\{\mathcal{Z}_t\right\}_p U_{-\nu} \tag{6.29}$$
$$A_1(z,\epsilon) \;:=\; U_{\nu} \Big(\left\{\mathcal{R}_{ex}(z)\right\}_p + \left\{\mathcal{R}^{\tau}_{ex}(z)\right\}_p + \left\{\mathcal{R}^{\tau}_{ld}(z)\right\}_p - \lambda^2 \mathrm{i} \left\{\mathrm{ad}(\varepsilon(P))\right\}_p \Big) U_{-\nu} \tag{6.30}$$
$$N \;:=\; U_{\nu} \left\{\mathcal{M}\right\}_p U_{-\nu} \tag{6.31}$$
$$B \;:=\; -U_{\nu} \left\{\mathrm{ad}(Y)\right\}_p U_{-\nu} \simeq -\mathrm{ad}(Y) \tag{6.32}$$
and
$$(p,\nu) \in \mathfrak{D}^{low}_{rw} \cap \left\{ |\mathrm{Im}\, p| \leq \min(\delta_{ex}, \delta_{\varepsilon}), \mathrm{Im}\, \nu \leq \frac{1}{2} \min(\delta_{ex}, \delta_{\varepsilon}) \right\} \tag{6.33}$$

Hence, the only difference with the relations (6.9-6.10-6.11-6.12) is that we have added the term $\left\{\mathcal{R}_{ex}(z)\right\}_p$ in (6.30), we consider $\mathcal{Z}_t$ instead of $\mathcal{Z}^{\tau}_t$ in (6.29), and we replace δ_1 by δ_{ex} in (6.33). This means that we can copy the proof of Lemma 6.2, except that, in addition, we have to check the bounds (B.6) and (B.7) for the term $\mathcal{R}_{ex}(z)$. We choose $g_A := 1/2 g_{ex}$. Then the bound (B.6) follows from (6.28), and (B.7) follows since, by the Cauchy integral formula and (6.28),
$$\sup_{\mathrm{Re}\, z \geq -\frac{1}{2}\lambda^2 g_{ex}} \| \frac{\partial}{\partial z} U_{\nu} \left\{\mathcal{R}_{ex}(z)\right\}_p U_{-\nu}\| = \left|\mathrm{Re}\, z - (-\lambda^2 g_{ex})\right|^{-1} o(\lambda^2) = o(|\lambda|^0), \qquad \lambda \searrow 0 \tag{6.34}$$
where we use the same argument as in (6.15), but with a circle radius equal to $\frac{1}{2}\left|\mathrm{Re}\, z - (-\lambda^2 g_{ex})\right|$. This application of the Cauchy integral formula is the reason for the factor $\frac{1}{2}$ into the definition of g_A. Lemma B.1 yields the function $f(p,\lambda)$, the rank-one projector $P(p,\lambda)$ and the operator $R^{low}(t,p,\lambda)$ required in the small fiber statements of Theorem 3.3.

For
$$(p,\nu) \in \mathfrak{D}^{high}_{rw} \cap \left\{ |\mathrm{Im}\, p| \leq \min(\delta_{ex}, \delta_{\varepsilon}), \mathrm{Im}\, \nu \leq \frac{1}{2} \min(\delta_{ex}, \delta_{\varepsilon}) \right\}, \tag{6.35}$$
we can again apply Lemma B.2 to derive the large fiber statements of Theorem 3.3. As in the proof of Lemma 6.2, we can again choose parameters δ, p^* such that domains $\mathfrak{D}^{low}, \mathfrak{D}^{high}$ as defined in (3.15-3.16), are included in the domains for (p,ν) specified by (6.33) and (6.35). □

7 Feynman Diagrams

In this section, we introduce the expansion of the reduced evolution $\mathcal{Z}_t$ and the cutoff reduced evolution $\mathcal{Z}^{\tau}_t$ in amplitudes labelled by Feynman diagrams. These expansions will be the main tool in the proofs of Lemmata 6.1 and 6.5. We start by introducing a notation for the Dyson expansion of $\mathcal{Z}_t$ which is more convenient than that of Section 5.2.

7.1 Diagrams σ

Consider a pair of elements in $I \times \mathbb{Z}^d \times \{L, R\}$ with $I \subset \mathbb{R}^+$ a closed interval whose elements should be thought of as times. The smaller time of the pair is called u and the larger time is called v, and we require that $u \neq v$, i.e. $u < v$. The set of pairs satisfying this constraint is called Σ_I^1. We define Σ_I^n as the set of n pairs of elements in $I \times \mathbb{Z}^d \times \{L, R\}$ such that no two times coincide. That is, each $\sigma \in \Sigma_I^n$ consists of n pairs whose time-coordinates are parametrized by (u_i, v_i), for $i = 1, \ldots, n$, and with the convention that $u_i < v_i$ and $u_i < u_{i+1}$. The elements σ are called *diagrams*. As announced in Section 5.2, there is a one-to-one mapping between, on the one hand, a set of triples $(t_i(\sigma), x_i(\sigma), l_i(\sigma))_{i=1}^{2n}$ with $t_i < t_{i+1}$ and $t_i \in I$, together with a pairing $\pi \in \mathcal{P}_n$, and, on the other hand, a diagram $\sigma \in \Sigma_I^n$ as defined above.
To construct this mapping, proceed as follows: Choose from the pairing π the pair (r, s) for which $r = 1$ and set $u_1 = t_r, v_1 = t_s$. The pair $((t_r, x_r, l_r), (t_s, x_s, l_s))$ becomes the first pair in the diagram σ. Then choose the pair $(r', s') \in \pi$ such that

$$r' = \min\{\{1, 2, \ldots, 2n\} \setminus \{r, s\}\} \tag{7.1}$$

Set $u_2 = t_{r'}, v_2 = t_{s'}$. The pair $((t_{r'}, x_{r'}, l_{r'}), (t_{s'}, x_{s'}, l_{s'}))$ becomes the second pair of σ. Repeat this until one has n pairs, each time picking the pair whose r is the smallest of the remaining integers. The mapping is easily visualized in a picture, see Fig. 7.

We also use the notation $\underline{t}(\sigma), \underline{x}(\sigma), \underline{l}(\sigma)$ to denote the 'coordinates' of the diagram σ. Here, $\underline{t}(\sigma), \underline{x}(\sigma), \underline{l}(\sigma)$ are $2n$-tuples of elements in $I, \mathbb{Z}^d, \{L, R\}$, respectively, and such that the i'th component of these $2n$-tuples constitutes the i'th triple $(t_i(\sigma), x_i(\sigma), l_i(\sigma))$.

Note that the time-coordinates $\underline{t} \equiv t_1(\sigma), \ldots, t_{2n}(\sigma)$ can also be defined as the ordered set of times containing the elements $\{u_i, v_i, i = 1, \ldots, n\}$. Evidently, the triples $(t_i(\sigma), x_i(\sigma), l_i(\sigma))_{i=1}^{2n}$ do not fix a diagram uniquely since the combinatorial structure that is encoded in π is missing. That combinatorial structure is now encoded in the way the time coordinates $\underline{t}(\sigma)$ are partitioned into pairs (u_i, v_i), see also Figure 7.

We drop the superscript n to denote the union over all $n \geq 1$, i.e.

$$\Sigma_I := \bigcup_{n \geq 1} \Sigma_I^n \tag{7.2}$$

and we write $|\sigma| = n$ to denote that $\sigma \in \Sigma_I^n$.

We define the *domain* of a diagram as

$$\mathrm{Dom}\sigma := \bigcup_{i=1}^n [u_i, v_i], \qquad \text{for } \sigma \in \Sigma_I^n \tag{7.3}$$

We call a diagram $\sigma \in \Sigma_I$ irreducible (notation: ir) whenever its domain is a connected set (hence an interval). In other words, σ is irreducible whenever there are no two (sub)diagrams $\sigma_1, \sigma_2 \in \Sigma_I$ such that

$$\sigma = \sigma_1 \cup \sigma_2, \qquad \text{and} \qquad \mathrm{Dom}\sigma_1 \cap \mathrm{Dom}\sigma_2 = \emptyset \tag{7.4}$$

where the union refers to a union of pairs of elements in $I \times \mathbb{Z}^d \times \{L, R\}$. For any $\sigma \in \Sigma_I$ that is not irreducible, we can thus find a unique (up to the order) sequence of (sub) diagrams $\sigma_1, \ldots, \sigma_m$ such that

$$\sigma_1, \ldots, \sigma_m \quad \text{are irreducible and} \quad \sigma = \sigma_1 \cup \ldots \cup \sigma_m \tag{7.5}$$

We fix the order of $\sigma_1, \ldots, \sigma_m$ by requiring that $\max \mathrm{Dom}\sigma_i \leq \min \mathrm{Dom}\sigma_{i+1}$ and we call the sequence $(\sigma_1, \ldots, \sigma_m)$ obtained in this way the decomposition of σ into irreducible components.

We let $\Sigma_I^n(\mathrm{ir}) \subset \Sigma_I^n$ stand for the set of irreducible diagrams σ (with n pairs) that satisfy $\mathrm{Dom}\sigma = I$, that is, $u_1 = t_1(\sigma) = \min I$ and $\max_i t_i(\sigma) = \max_i v_i = \max I$.

A diagram $\sigma \in \Sigma_I(\mathrm{ir})$ is called *minimally irreducible* in the interval I whenever it has the following property: For any subdiagram $\sigma' \subset \sigma$, the diagram $\sigma \setminus \sigma'$ does not belong to $\Sigma_I(\mathrm{ir})$. Intuitively, this means that either the

diagram σ' contains a boundary point of I as one of its time-coordinates, or the diagram $\sigma \setminus \sigma'$ is not irreducible. The set of minimally irreducible diagrams (with n pairs) is denoted by $\Sigma_I^n(\mathrm{mir})$. See pictures 7 and 8 for a graphical representation of the diagrams. Since, up to now, most definitions depend solely on the time-coordinates, we only indicate the time-coordinates in the pictures. In the terminology introduced below, we draw equivalence classes of diagrams $[\sigma]$ rather than the diagrams σ themselves.

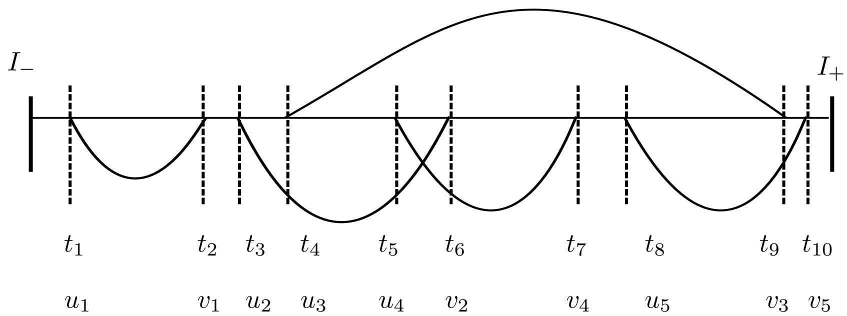


Figure 7: A diagram $\sigma \in \Sigma_I$ with $|\sigma| = 5$. Its time coordinates are shown explicitly. Note that the parametrization by u_i, v_i encodes the combinatorial structure (the way the times are connected by pairings), whereas the t_i are ordered. We consistenly draw the long pairings (see later) above the time-axis and the short ones below.

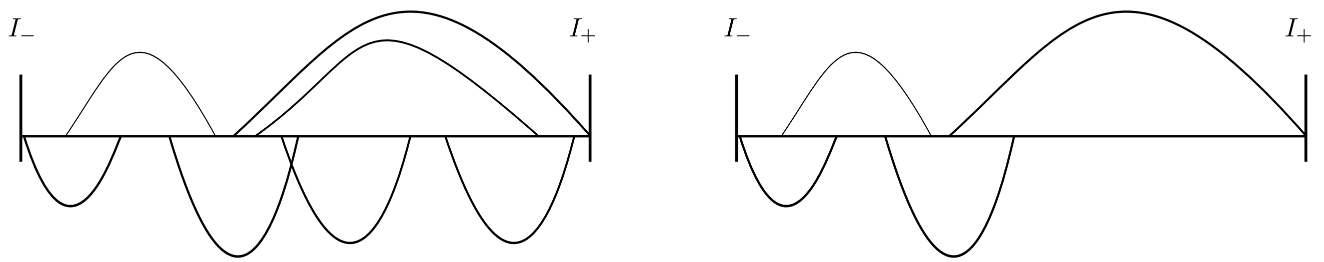


Figure 8: The left figure shows an irreducible diagram σ in the interval $I = [I_-, I_+]$ with $|\sigma| = 7$. This diagram is not minimally irreducible. The right figure shows a minimally irreducible subdiagram. In this case, there is only one such minimally irreducible subdiagram, but this need not always be the case.

A diagram σ in Σ_I for which each pair of time coordinates (u, v) satisfies $|v - u| \geq \tau$, or $|v - u| \leq \tau$, is called long, or short, respectively. The set of all long/small diagrams with n pairs is denoted by $\Sigma_I^n(> \tau)$ / $\Sigma_I^n(< \tau)$. Note that $\Sigma_I^n(> \tau) \cup \Sigma_I^n(< \tau)$ is strictly smaller than Σ_I^n whenever $n > 1$.

In addition to the sets $\Sigma_I^n(\mathrm{ir}), \Sigma_I^n(\mathrm{mir}), \Sigma_I^n(> \tau)$, we will sometimes use more than one specification (adj) to denote a subset of Σ_I or Σ_I^n, and we will drop the superscript n to denote the union over all $|\sigma|$, as in (7.3), for example,

$$\Sigma_I^n(< \tau, \mathrm{ir}), \qquad \Sigma_I(> \tau, \mathrm{mir}) \tag{7.6}$$

are the sets of short irreducible diagrams with $|\sigma| = n$ and long minimally irreducible diagrams, respectively.

On the set Σ_I^n, we define the "Lebesgue measure" $\mathrm{d}\sigma$ by

$$\int_{\Sigma_I^n} \mathrm{d}\sigma F(\sigma) := \int_{I_- < u_1 < \ldots < u_n < I_+} \mathrm{d}u_1 \ldots \mathrm{d}u_n \int_{u_i < v_i} \mathrm{d}v_1 \ldots \mathrm{d}v_n \sum_{\underline{x}(\sigma), \underline{l}(\sigma)} F(\sigma) \tag{7.7}$$

where $I = [I_-, I_+]$.

Since $\Sigma_I^n(\mathrm{ir})$ is a zero-measure subset of Σ_I^n, the definition of the measure $\mathrm{d}\sigma$ on $\Sigma_I^n(\mathrm{ir})$ has to modified in an

obvious way: For all continuous (in the time coordinates $\underline{t}(\sigma)$) functions F on Σ_I^n, we set

$$\int\limits_{\Sigma_I^n(\mathrm{ir})} \mathrm{d}\sigma F(\sigma) = \int\limits_{\Sigma_I^n} \mathrm{d}\sigma \delta(\max \underline{t}(\sigma) - I_+)\delta(\min \underline{t}(\sigma) - I_-)F(\sigma) \tag{7.8}$$

where the Dirac distributions $\delta(I_+ - \cdot)$ and $\delta(I_- - \cdot)$ are a priori ambiguous since I_-, I_+ are the boundary points of the interval I. They are defined as

$$\delta(\cdot - I_+) := \lim_{\substack{s \to I_+ \\ <}} \delta(\cdot - s), \qquad \delta(\cdot - I_-) := \lim_{\substack{s \to I_- \\ >}} \delta(\cdot - s) \tag{7.9}$$

We extend the definition of the measure $\mathrm{d}\sigma$ also to Σ_I and the various $\Sigma_I(\mathrm{adj})$ by setting

$$\int\limits_{\Sigma_I(\mathrm{adj})} \mathrm{d}\sigma F(\sigma) := \sum_{n \geq 1} \int\limits_{\Sigma_I^n(\mathrm{adj})} \mathrm{d}\sigma F_n(\sigma) \tag{7.10}$$

where F_n is the restriction to $\Sigma_I^n(\mathrm{adj})$ of a function F on $\Sigma_I(\mathrm{adj})$.

We will often encounter functions of σ that are independent of the coordinates $\underline{x}(\sigma), \underline{l}(\sigma)$ and that must be integrated only over $\underline{t}(\sigma)$ and summed over $|\sigma|$. To deal elegantly with such situations, we let $[\sigma]$ stand for an equivalence class of diagrams that is obtained by dropping the $\underline{x}, \underline{l}$-coordinates. That is

$$[\sigma] = [\sigma'] \Leftrightarrow \begin{cases} |\sigma| = |\sigma'| \\ u_i(\sigma) = u_i(\sigma'), v_i(\sigma) = v_i(\sigma'), \text{ for all } i = 1, \dots, |\sigma| \end{cases} \tag{7.11}$$

The set of such equivalence classes is denoted by $\Pi_T \Sigma_I$ (the symbol Π_T can be thought of as a projection onto the time coordinates) and we naturally extend the definition to $\Pi_T \Sigma_I(\mathrm{adj})$ where adj can again stand for $\mathrm{ir}, \mathrm{mir}, > \tau, < \tau$. The integration over equivalence classes of diagrams is defined as above in (7.7) and (7.8), but with $\sum_{\underline{x}(\sigma), \underline{l}(\sigma)}$ omitted, i.e., such that for all functions $\tilde{F}$ on $\Sigma_I(\mathrm{adj})$:

$$\int\limits_{\Pi_T \Sigma_I(\mathrm{adj})} \mathrm{d}[\sigma] F([\sigma]) = \int\limits_{\Sigma_I(\mathrm{adj})} \mathrm{d}\sigma \tilde{F}(\sigma), \qquad \text{with } F([\sigma]) = \sum_{\underline{x}(\sigma), \underline{l}(\sigma)} \tilde{F}(\sigma) \tag{7.12}$$

Lemma 7.1 contains the main application of this construction. It is in fact a simple $L^1 - L^\infty$-bound.

Lemma 7.1. *Let F and G be positive, continuous functions on Σ_I. Then*

$$\int\limits_{\Sigma_I(\mathrm{adj})} \mathrm{d}\sigma F(\sigma) G(\sigma) \leq \int\limits_{\Pi_T \Sigma_I(\mathrm{adj})} \mathrm{d}[\sigma] \left[\sup_{\underline{x}(\sigma), \underline{l}(\sigma)} G(\sigma) \right] \left[\sup_{\underline{t}(\sigma)} \sum_{\underline{x}(\sigma), \underline{l}(\sigma)} F(\sigma) \right] \tag{7.13}$$

where it is understood that the sum and sup over $\underline{x}(\sigma), \underline{l}(\sigma)$ are performed while keeping $|\sigma|$ and $\underline{t}(\sigma)$ fixed.

In (7.13), the sum/sup, over $\underline{x}(\sigma), \underline{l}(\sigma)$ is in fact a shorthand notation for the sum/sup over all σ' such that $[\sigma'] = [\sigma]$ for a given σ. Hence, $\sup_{\underline{x}(\sigma), \underline{l}(\sigma)} G(\sigma)$ is a function of $[\sigma]$ only, as required. The supremum $\sup_{\underline{t}(\sigma)}$ is over $I^{2|\sigma|}$, with $|\sigma|$ fixed. Hence, the second factor on the RHS of (7.13) is in fact a function of $|\sigma|$ only.

Proof. We start from the explicit expressions in (7.7) or (7.8), and we use a $L^1 - L^\infty$ inequality: first for the sum over $\underline{x}(\sigma), \underline{l}(\sigma)$ and then for the integration over u_i, v_i. $\square$

7.1.1 Representation of the reduced evolution $\mathcal{Z}_t$

Recall the operators $\mathcal{V}_t((t_i,x_i,l_i)_{i=1}^{2n})$ defined in (5.9). Since, by the above discussion, there is a one-to-one correspondence between a diagram σ and $(\pi,(t_i,x_i,l_i)_{i=1}^{2|\sigma|})$ where $\pi\in\mathcal{P}_{|\sigma|}$ and $t_i<t_{i+1}$, we can write $\mathcal{V}_t(\sigma)$ instead of $\mathcal{V}_t((t_i,x_i,l_i)_{i=1}^{2|\sigma|})$ and $\zeta(\sigma)$ instead of $\zeta(\pi,(t_i,x_i,l_i)_{i=1}^{2|\sigma|})$, i.e.

$$\mathcal{V}_t(\sigma):=\mathcal{V}_t((t_i(\sigma),x_i(\sigma),l_i(\sigma))_{i=1}^{2|\sigma|}) \tag{7.14}$$

and

$$\zeta(\sigma):=\prod_{((u,x,l),(v,x',l'))\in\sigma}\lambda^2\begin{cases}\psi(x'-x,v-u) & l=l'=L\\ \overline{\psi}(x'-x,v-u) & l=l'=R\\ \overline{\psi}(x'-x,v-u) & l=L,l'=R\\ \psi(x'-x,v-u) & l=R,l'=L\end{cases} \tag{7.15}$$

As a slight generalization of the operators $\mathcal{V}_t(\sigma)$, we also define $\mathcal{V}_I(\sigma)$ for a closed interval $I:=[I_-,I_+]$ by

$$\mathcal{V}_I(\sigma):=\mathcal{U}_{I_+-t_{2n}}\mathcal{I}_{x_{2n},l_{2n}}\ldots\mathcal{I}_{x_2,l_2}\mathcal{U}_{t_2-t_1}\mathcal{I}_{x_1,l_1}\mathcal{U}_{t_1-I_-},\qquad \text{for } \sigma \text{ such that } \mathrm{Dom}\sigma\subset I \tag{7.16}$$

The only difference with $\mathcal{V}_t(\sigma)$ is in the time-arguments 't' of $\mathcal{U}_t$ at the beginning and the end of the expression. With this new notation, $\mathcal{V}_t(\sigma)=\mathcal{V}_{[0,t]}(\sigma)$. Next, we state the representation of the reduced evolution $\mathcal{Z}_t$ as an integral over diagrams

$$\mathcal{Z}_t=\mathcal{U}_t+\int_{\Sigma_{[0,t]}}\mathrm{d}\sigma\zeta(\sigma)\mathcal{V}_{[0,t]}(\sigma) \tag{7.17}$$

Similarly, the cutoff dynamics $\mathcal{Z}_t^\tau$ is represented as

$$\mathcal{Z}_t^\tau=\mathcal{U}_t+\int\limits_{\Sigma_{[0,t]}(<\tau)}\mathrm{d}\sigma\zeta(\sigma)\mathcal{V}_{[0,t]}(\sigma) \tag{7.18}$$

Formulas (7.17) and (7.18) are immediate consequences of (5.8) and (5.15), respectively.

We use the notion of irreducible diagrams σ to decompose the operators $\mathcal{V}_{[0,t]}(\sigma)$ into products and to derive a new representation, (7.23), for $\mathcal{Z}_t$ and $\mathcal{Z}_t^\tau$

Let $(\sigma_1,\ldots,\sigma_p)$ be the decomposition of a diagram $\sigma\in\Sigma_{[0,t]}$ into irreducible components. Define the times $s_1,\ldots,s_{2p}$ to be the boundaries of the domains of the irreducible components, i.e., $[s_{2i-1},s_{2i}]=\mathrm{Dom}\sigma_i$, for $i=1,\ldots,p$. Then

$$\mathcal{V}_I(\sigma)=\mathcal{U}_{I_+-s_{2p}}\,\mathcal{V}_{[s_{2p-1},s_{2p}]}(\sigma_p)\,\mathcal{U}_{s_{2p-1}-s_{2p-2}}\ldots\mathcal{U}_{s_3-s_2}\,\mathcal{V}_{[s_1,s_2]}(\sigma_1)\,\mathcal{U}_{s_1-I_-}, \tag{7.19}$$

as can be checked from (7.15-7.16). Here, the essential observation is that all time coordinates of σ_i are smaller than those of σ_{i+1}. We introduce

$$\mathcal{Z}_t^{\mathrm{ir}}:=\int\limits_{\Sigma_{[0,t]}(\mathrm{ir})}\mathrm{d}\sigma\zeta(\sigma)\mathcal{V}_{[0,t]}(\sigma),\qquad \mathcal{Z}_t^{\tau,\mathrm{ir}}:=\int\limits_{\Sigma_{[0,t]}(<\tau,\mathrm{ir})}\mathrm{d}\sigma\zeta(\sigma)\mathcal{V}_{[0,t]}(\sigma) \tag{7.20}$$

and we remark that the definitions in (7.20) allow for a shift of time on the RHS, that is

$$\mathcal{Z}_t^{\mathrm{ir}}=\int\limits_{\Sigma_I(\mathrm{ir})}\mathrm{d}\sigma\zeta(\sigma)\mathcal{V}_I(\sigma),\qquad \text{for any } I=[s,s+t],\quad s\in\mathbb{R} \tag{7.21}$$

and similarly for $\mathcal{Z}_t^{\tau,\mathrm{ir}}$. By this time-translation invariance, the factorization property (7.19) and the factorization property of the correlation function in (7.15), i.e.,

$$\zeta(\sigma_1\cup\ldots\cup\sigma_p)=\prod_{i=1}^p\zeta(\sigma_i), \tag{7.22}$$

we can rewrite the expression (7.17) as

$$\mathcal{Z}_t = \sum_{m\in 2\mathbb{Z}^+} \int\limits_{0\leq s_1\leq \ldots \leq s_m \leq t} \mathrm{d}s_1 \ldots \mathrm{d}s_m \, \left(\mathcal{U}_{t-s_m} \mathcal{Z}^{\mathrm{ir}}_{s_m - s_{m-1}} \ldots \mathcal{U}_{s_3-s_2} \mathcal{Z}^{\mathrm{ir}}_{s_2-s_1} \mathcal{U}_{s_1} \right). \tag{7.23}$$

where the term on the RHS corresponding to $m=0$ is understood to be equal to $\mathcal{U}_t$. The idea behind (7.23) is that, instead of summing over all diagrams, we sum over all sequences of irreducible diagrams. An analogous formula holds with $\mathcal{Z}_t$ and $\mathcal{Z}_t^{\mathrm{ir}}$ replaced by $\mathcal{Z}_t^{\tau}$ and $\mathcal{Z}_t^{\tau,\mathrm{ir}}$.

7.2 Ladder diagrams and excitations

We are ready to identify the operators $\mathcal{R}^{\tau}_{ex}(z)$ and $\mathcal{R}^{\tau}_{\mathrm{ld}}(z)$, whose existence was postulated in Lemma 6.1 and the operator $\mathcal{R}_{ex}(z)$, which was postulated in Lemma 6.5.

The Laplace transform, $\mathcal{R}(z)$, of $\mathcal{Z}_t$ has been introduced in (6.2). We calculate $\mathcal{R}(z)$ starting from (7.23)

$$\mathcal{R}(z) = \int_{\mathbb{R}^+} \mathrm{d}t \, \mathrm{e}^{-tz} \mathcal{Z}_t \tag{7.24}$$

$$= \sum_{m\geq 0} \left[(z + \mathrm{i}\mathrm{ad}(H_{\mathrm{S}}))^{-1} \int_{\mathbb{R}^+} \mathrm{d}t \, \mathrm{e}^{-tz} \mathcal{Z}_t^{\mathrm{ir}} \right]^m (z + \mathrm{i}\mathrm{ad}(H_{\mathrm{S}}))^{-1} \tag{7.25}$$

$$= (z + \mathrm{i}\mathrm{ad}(H_{\mathrm{S}}) - \mathcal{R}_{\mathrm{ir}}(z))^{-1}, \qquad \text{with } \mathcal{R}_{\mathrm{ir}}(z) := \int_{\mathbb{R}^+} \mathrm{d}t \, \mathrm{e}^{-tz} \mathcal{Z}_t^{\mathrm{ir}} \tag{7.26}$$

The second equality follows by $\int_{\mathbb{R}^+} \mathrm{d}t \mathrm{e}^{-tz} \mathcal{U}_t = (z + \mathrm{i}\mathrm{ad}(H_{\mathrm{S}}))^{-1}$ for $\mathrm{Re}\, z > 0$. The third equality follows by summing the geometric series.

An identical computation yields

$$\mathcal{R}^{\tau}(z) = (z + \mathrm{i}\mathrm{ad}(H_{\mathrm{S}}) - \mathcal{R}^{\tau}_{\mathrm{ir}}(z))^{-1}, \qquad \text{with } \mathcal{R}^{\tau}_{\mathrm{ir}}(z) := \int_{\mathbb{R}^+} \mathrm{d}t \, \mathrm{e}^{-tz} \mathcal{Z}_t^{\tau,\mathrm{ir}} \tag{7.27}$$

The definition of $\mathcal{R}^{\tau}_{ex}(z)$ and $\mathcal{R}^{\tau}_{\mathrm{ld}}(z)$ relies on the following splitting of $\mathcal{R}^{\tau}_{\mathrm{ir}}(z)$

$$\mathcal{R}^{\tau}_{\mathrm{ld}}(z) := \int\limits_{\mathbb{R}^+} \mathrm{d}t \, \mathrm{e}^{-tz} \int\limits_{\Sigma_{[0,t]}(<\tau,\mathrm{ir})} \zeta(\sigma) 1_{|\sigma|=1} \mathcal{V}_{[0,t]}(\sigma) \tag{7.28}$$

$$\mathcal{R}^{\tau}_{ex}(z) := \int\limits_{\mathbb{R}^+} \mathrm{d}t \, \mathrm{e}^{-tz} \int\limits_{\Sigma_{[0,t]}(<\tau,\mathrm{ir})} \zeta(\sigma) 1_{|\sigma|\geq 2} \mathcal{V}_{[0,t]}(\sigma) \tag{7.29}$$

The subscripts refer to ''ladder'- and 'excitation'-diagrams. The name 'ladder' originates from the graphical representation of diagrams whose irreducible components consist of one pair (it is standard in condensed matter theory). Since obviously $\mathcal{R}^{\tau}_{ex}(z) + \mathcal{R}^{\tau}_{\mathrm{ld}}(z) = \mathcal{R}^{\tau}_{\mathrm{ir}}(z)$, the relation (7.27) implies Statement (1) of Lemma 6.1.

In the model without cutoff, we do not disentangle ladder and excitation diagrams, since every diagram that contains a long pairing, is considered an excitation. We can thus define

$$\mathcal{R}_{ex}(z) := \mathcal{R}_{\mathrm{ir}}(z) - \mathcal{R}^{\tau}_{\mathrm{ir}}(z) \tag{7.30}$$

We will come up with a more constructive representation of $\mathcal{R}_{ex}(z)$ in formula (7.33).

7.2.1 The reduced evolution as a a double integral over long and short diagrams

We develop a new representation of $\mathcal{Z}_t^{\mathrm{ir}}$ by fixing the long diagrams, i.e., those in $\Sigma_{[0,t]}(>\tau)$, and integrating the short ones.

We define the *conditional cutoff dynamics*, $\mathcal{C}_t(\sigma_l)$, depending on a long diagram $\sigma_l \in \Sigma_{[0,t]}(>\tau)$, as follows:

$$\mathcal{C}_t(\sigma_l) = 1_{\sigma_l \in \Sigma_{[0,t]}(>\tau,\mathrm{ir})} \mathcal{V}_{[0,t]}(\sigma_l) + \int\limits_{\substack{\Sigma_{[0,t]}(<\tau) \\ \sigma_l \cup \sigma \in \Sigma_{[0,t]}(\mathrm{ir})}} \mathrm{d}\sigma\, \zeta(\sigma) \mathcal{V}_{[0,t]}(\sigma \cup \sigma_l) \tag{7.31}$$

In words, $\mathcal{C}_t(\sigma_l)$ contains contributions of short diagrams $\sigma \in \Sigma_t(<\tau)$ such that $\sigma_l \cup \sigma$ is irreducible in the interval $[0,t]$. Hence, if σ_l is itself irreducible in the interval $[0,t]$, then there is a term without any short diagrams; this is the first term in (7.31). In general, σ_l need not be irreducible. Note that the constraint on σ (in the domain of the integral) in the second term of (7.31) depends crucially on the nature of σ_l. In particular, if $\mathrm{Dom}\sigma_l$ does not contain the boundary points 0 or t, then σ has to contain 0 or t, and this introduces one or two delta functions into the constraint on σ. To relate $\mathcal{C}_t(\sigma_l)$ to $\mathcal{Z}_t^{\mathrm{ir}}$, we must explicitly add those σ_l that contain one or both of the times 0 and t. This is visible in the following formula, which follows from (7.31) and the definition of $\mathcal{Z}_t^{\mathrm{ir}}$ in (7.20).

$$\mathcal{Z}_t^{\mathrm{ir}} - \mathcal{Z}_t^{\tau,\mathrm{ir}} = \int\limits_{\Sigma_{[0,t]}(>\tau)} \mathrm{d}\sigma_l \zeta(\sigma_l) \mathcal{C}_t(\sigma_l)\, \left[1 + \delta(t_1(\sigma_l))\right] \left[1 + \delta(t_{2|\sigma_l|}(\sigma_l) - t)\right] \tag{7.32}$$

We must subtract $\mathcal{Z}_t^{\tau,\mathrm{ir}}$ on the LHS, since all contributions to the RHS involve at least one long diagram. The δ-functions on the RHS are defined as in (7.9).

The following formula is an obvious consequence of (7.30) and (7.32):

$$\mathcal{R}_{ex}(z) = \int\limits_{\mathbb{R}^+} \mathrm{d}t\, \mathrm{e}^{-tz} \int\limits_{\Sigma_{[0,t]}(>\tau)} \mathrm{d}\sigma_l \zeta(\sigma_l) \mathcal{C}_t(\sigma_l)\, \left[1 + \delta(t_1(\sigma_l))\right] \left[1 + \delta(t_{2|\sigma_l|}(\sigma_l) - t)\right] \tag{7.33}$$

All of Section 8 will be devoted to proving good bounds on $\mathcal{R}_{ex}(z)$, as claimed in Lemma 6.5.

7.3 Decomposition of the conditional cutoff dynamics $\mathcal{C}_t(\sigma_l)$

Our next step is to decompose the conditional cutoff dynamics $\mathcal{C}_t(\sigma_l)$, as defined in (7.31), into components. Since $\mathcal{C}_t(\sigma_l)$ is defined as an integral over short diagrams σ, we can achieve this by classifying the short diagrams σ that contribute to this integral. The idea is to look at the irreducible components of σ whose domain contains one or more of the time-coordinates of σ_l (In our final formula, (7.45), these domains correspond to the intervals $[s_k^{\mathrm{i}}, s_k^{\mathrm{f}}]$). The irreducible components whose domain does not contain any of the time coordinates of σ_l can be resummed right away, and they do not play a role in our classification (this corresponds to the operators $\mathcal{Z}_t^{\tau}$ in (7.45)). We outline the abstract decomposition procedure in Section 7.3.1, and we present an example (with figures) in Section 7.3.2.

7.3.1 Vertices and vertex partitions

Consider a long diagram $\sigma_l \in \Sigma_{[0,t]}(>\tau)$ with $|\sigma_l| = n$ and time-coordinates $\underline{t}(\sigma_l) = (t_1, \ldots, t_{2n})$. With this diagram, we will associate different *vertex partitions* $\mathfrak{L}$. First, we define *vertices*. A vertex $\mathfrak{l}$ is determined by a label, *bare* or *dressed*, and a vertex set $S(\mathfrak{l})$, given by

$$S(\mathfrak{l}) = \{t_j, t_{j+1} \ldots, t_{j+m-1}\}, \qquad \text{for some } 1 \leq j < j+m-1 \leq 2n \tag{7.34}$$

Hence, the vertex set is a a subset of the times $\{t_1(\sigma_l), \ldots, t_{2n}(\sigma_l)\}$. Moreover, a vertex $\mathfrak{l}$ with $|S(\mathfrak{l})| > 1$ is always dressed. Hence, if $\mathfrak{l}$ is bare then $S(\mathfrak{l})$ is necessarily a singleton, i.e., $S(\mathfrak{l}) = \{t_j\}$ for some j.

A vertex partition $\mathfrak{L}$ compatible with σ_l (Notation: $\mathfrak{L} \sim \sigma_l$) is a collection of vertices $\mathfrak{l}_1, \ldots, \mathfrak{l}_m$ such that

- The vertex sets $S(\mathfrak{l}_1), \ldots, S(\mathfrak{l}_p)$ form a partition of $\{t_1(\sigma_l), \ldots, t_{2n}(\sigma_l)\}$. By convention, we always number the vertices in a vertex partition such that the elements of $S(\mathfrak{l}_k)$ are smaller than those of $S(\mathfrak{l}_{k+1})$. The number, p, of vertices in a vertex partition is called the cardinality of the vertex partition and is denoted by $|\mathfrak{L}|$.

- Any two consecutive times t_j, t_{j+1} such that $[t_j, t_{j+1}] \not\subset \mathrm{Dom}\sigma_l$, belong to the vertex set $S(\mathfrak{l}_k)$ of one of the vertices $\mathfrak{l}_k$. Such a vertex $\mathfrak{l}_k$ is necessarily dressed since its vertex set contains at least two elements.
- If $t_1 = 0$, then $S(\mathfrak{l}_1) = \{t_1\}$ and $\mathfrak{l}_1$ is bare. If $t_1 > 0$, then $S(\mathfrak{l}_1) \ni t_1$ and $\mathfrak{l}_1$ is dressed.
- If $t_{2n} = t$, then $S(\mathfrak{l}_m) = \{t_{2n}\}$ and $\mathfrak{l}_m$ is bare. If $t_{2n} < t$, then $S(\mathfrak{l}_m) \ni t_{2n}$ and $\mathfrak{l}_m$ is dressed.

The idea is to split

$$\mathcal{C}_t(\sigma_l) = \sum_{\mathfrak{L} \sim \sigma_l} \mathcal{C}_t(\sigma_l, \mathfrak{L}) \tag{7.35}$$

where the sum is over all $\mathfrak{L}$ compatible with σ_l and $\mathcal{C}_t(\sigma_l, \mathfrak{L})$ contains the contributions of all short pairings σ that *match* the vertex partition $\mathfrak{L}$;

$$\sigma \text{ matches } \mathfrak{L} \quad \Leftrightarrow \quad \begin{cases} \forall \text{ dressed } \mathfrak{l}_k : \exists! \text{ irr. component } \sigma_j \subset \sigma \text{ such that } S(\mathfrak{l}_k) \subset \mathrm{Dom}\sigma_j \\ \text{and } S(\mathfrak{l}_{k'}) \cap \mathrm{Dom}\sigma_j = \emptyset \text{ for all } k' \neq k \\ \forall \text{ bare } \mathfrak{l}_k : S(\mathfrak{l}_k) \cap \mathrm{Dom}\sigma = \emptyset \end{cases} \tag{7.36}$$

For the sake of completeness, we define the operators $\mathcal{C}_t(\sigma_l, \mathfrak{L})$, below, but, in Section 7.3.3, we will provide a more constructive expression for them. First, assume that the vertex partition $\mathfrak{L}$ contains at least one dressed vertex. Then $\mathcal{C}_t(\sigma_l, \mathfrak{L})$ is defined by restricting the second integral in (7.31) to those σ that match $\mathfrak{L}$;

$$\mathcal{C}_t(\sigma_l, \mathfrak{L}) := \int_{\substack{\Sigma_{[0,t]}(<\tau) \\ \sigma_l \cup \sigma \in \Sigma_{[0,t]}(\mathrm{ir})}} \mathrm{d}\sigma \zeta(\sigma) \mathcal{V}_{[0,t]}(\sigma \cup \sigma_l) 1_{\sigma \text{ matches } \mathfrak{L}} \tag{7.37}$$

Next, we assume that the vertex partition $\mathfrak{L}$ contains only bare vertices. If σ_l is irreducible in the interval $[0, t]$, i.e., $\sigma_l \in \Sigma_{[0,t]}(> \tau, \mathrm{ir})$, then the vertex partition with only bare vertices is compatible with σ_l. If $\sigma_l \notin \Sigma_{[0,t]}(> \tau, \mathrm{ir})$, then this vertex partition is not compatible with σ_l. Hence, we assume that $\sigma_l \in \Sigma_{[0,t]}(> \tau, \mathrm{ir})$ and we define

$$\mathcal{C}_t(\sigma_l, \mathfrak{L}) := \mathcal{V}_{[0,t]}(\sigma_l) + \int_{\Sigma_{[0,t]}(<\tau)} \mathrm{d}\sigma \, \zeta(\sigma) \mathcal{V}_{[0,t]}(\sigma \cup \sigma_l) \times 1_{\{\{t_1, \ldots, t_{2|\sigma_l|}\} \cap \mathrm{Dom}\sigma = \emptyset\}} \tag{7.38}$$

The second term is the same as in (7.37) (but specialized to the partition with only bare vertices) and the first term is a contribution without any short diagrams. This first term equals the first term (on the RHS) of (7.31).

In Section 7.3.2, we give examples of vertex partitions that are intended to render the above concepts more intuitive.

7.3.2 Examples of vertex partitions

We choose a long diagram $\sigma_l \in \Sigma^3_{[0,t]}(> \tau)$ which consists of three pairs such that the time coordinates $(u_i, v_i)^3_{i=1}$ are ordered as

$$\begin{array}{ccccccccccccc} & & t_1 & & t_2 & & t_3 & & t_4 & & t_5 & & t_6 & & \\ 0 & < & u_1 & < & u_2 & < & v_1 & < & u_3 & < & v_2 & < & v_3 & = & t \end{array} \tag{7.39}$$

Hence, σ_l is irreducible in the interval $[u_1, t]$, but not in the interval $[0, t]$, at least not if $t_1 \neq 0$.

Below we display three diagrams $\sigma \in \Sigma_{[0,t]}(< \tau)$ satisfying the condition $\sigma_l \cup \sigma \in \Sigma_{[0,t]}(\mathrm{ir})$. To assign to each of those diagrams a vertex partition, we proceed as follows. Starting on the left, we look at the time-coordinates $\underline{t}(\sigma_l)$ and we check whether these times are 'bridged' by a short pairing, i.e., whether they belong to the domain of a short diagram. If this is the case then such a time belongs to the vertex set of a dressed vertex. The vertex set of this vertex is the set of all time-coordinates that are connected to this point by short pairings. If this is not the case, i.e., if a time-coordinate of σ_l is not 'bridged' by any short pairing, than such a point constitutes a bare vertex, whose vertex set is just this one point.

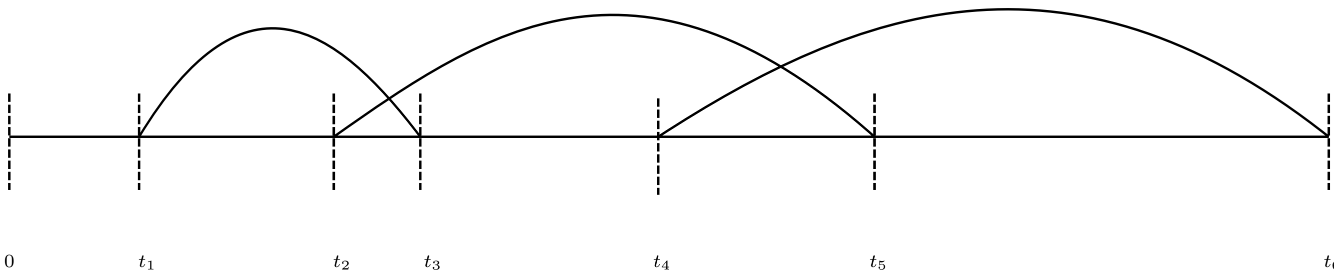


Figure 9: A long diagram with time coordinates as in (7.39)

Actually, for the first time-coordinate (in our case u_1), this is particularly simple. Either the first time-coordinate is **not** equal to 0, in which case it has to be 'connected' by short pairings to 0 (indeed, if this were not the case, then $\sigma_l \cup \sigma$ cannot be irreducible in $[0, t]$), or the first time-coordinate **is** equal to 0, in which case it cannot be connected by short pairings to the second coordinate, because then the first time-coordinate of the short diagram would have to be 0 as well, which is a zero measure event (for this reason, we have excluded this case in the definition of the diagrams in Section 7.1). In our example $u_1 \neq 0$, and one checks that, in all three choices of σ, there are short diagrams connecting u_1 and 0.

Let us determine the vertices in the three displayed figures

vertex partition 1

$\mathfrak{l}_1$	$\{t_1\}$	dressed
$\mathfrak{l}_2$	$\{t_2\}$	bare
$\mathfrak{l}_3$	$\{t_3\}$	bare
$\mathfrak{l}_4$	$\{t_4\}$	dressed
$\mathfrak{l}_5$	$\{t_5\}$	bare
$\mathfrak{l}_6$	$\{t_6\}$	bare

vertex partition 2

$\mathfrak{l}_1$	$\{t_1\}$	dressed
$\mathfrak{l}_2$	$\{t_2\}$	bare
$\mathfrak{l}_3$	$\{t_3, t_4, t_5\}$	dressed
$\mathfrak{l}_4$	$\{t_6\}$	bare

vertex partition 3

$\mathfrak{l}_1$	$\{t_1\}$	dressed
$\mathfrak{l}_2$	$\{t_2, t_3\}$	dressed
$\mathfrak{l}_3$	$\{t_4\}$	bare
$\mathfrak{l}_4$	$\{t_5\}$	bare
$\mathfrak{l}_5$	$\{t_6\}$	bare

In the example displayed above, it is also very easy to determine which vertex partitions $\mathfrak{L}$ are compatible with σ_l ($\mathfrak{L} \sim \sigma_l$). Apart from the fact that the vertex sets $S(\mathfrak{l}_k)$ of the vertices in $\mathfrak{L}$ have to form a partition of $\{t_1, \ldots, t_6\}$, we need that $\mathfrak{l}_1$ is dressed and $\mathfrak{l}_{|\mathfrak{L}|}$ (the last vertex in the partition) is bare.

To each vertex $\mathfrak{l}_k$ in the above examples, we can associate time coordinates s_k^{i} and s_k^{f} as the boundary times of the domains of irreducible diagrams bridging the times in the vertex. Eventually, we intend to fix a vertex partition and associated time coordinates $\underline{s}^i$ and $\underline{s}^f$ and to integrate over all short diagrams that are irreducible in the interval $[s^i, s^f]$. This integration gives rise to the vertex operators, see eq. (7.42). To illustrate this, we zoom in on a part of a long diagram, shown in Figure 11. A formal definition is given in the next section.

7.3.3 Abstract definition of the vertex operator

Let $\mathfrak{L}$ be a vertex partition compatible with σ_l, with vertices $\mathfrak{l}_k, k = 1, \ldots, |\mathfrak{L}|$. In what follows, we focus on one particular vertex $\mathfrak{l}_k$ which we assume first to be dressed. The vertex $\mathfrak{l}_k$ is assumed to have a vertex set $S(\mathfrak{l}_k) = \{t_j, t_{j+1}, \ldots, t_{j+m-1}\}$. This means in particular that the time-coordinate t_{j-1} belongs to the vertex set of the vertex $\mathfrak{l}_{k-1}$ (unless $j = 1$) and the time-coordinate t_{j+m} belongs to the vertex set of the vertex $\mathfrak{l}_{k+1}$ (unless $j+m-1 = 2|\sigma_l|$). We fix an initial time s_k^{i} and final time s_k^{f} such that

$$t_{j-1} \leq s_k^{\mathrm{i}} \leq t_j \leq t_{j+m-1} \leq s_k^{\mathrm{f}} \leq t_{j+m} \tag{7.40}$$

where it is understood that $t_{j-1} = 0$ if $j = 1$ and $t_{j+m} = t$ if $j + m - 1 = 2|\sigma_l|$. The vertex operator $\mathcal{B}(\mathfrak{l}_k, s_k^{\mathrm{i}}, s_k^{\mathrm{f}})$ is defined by summing the contributions of all $\sigma \in \Sigma_{[s_k^{\mathrm{i}}, s_k^{\mathrm{f}}]}(< \tau, \mathrm{ir})$

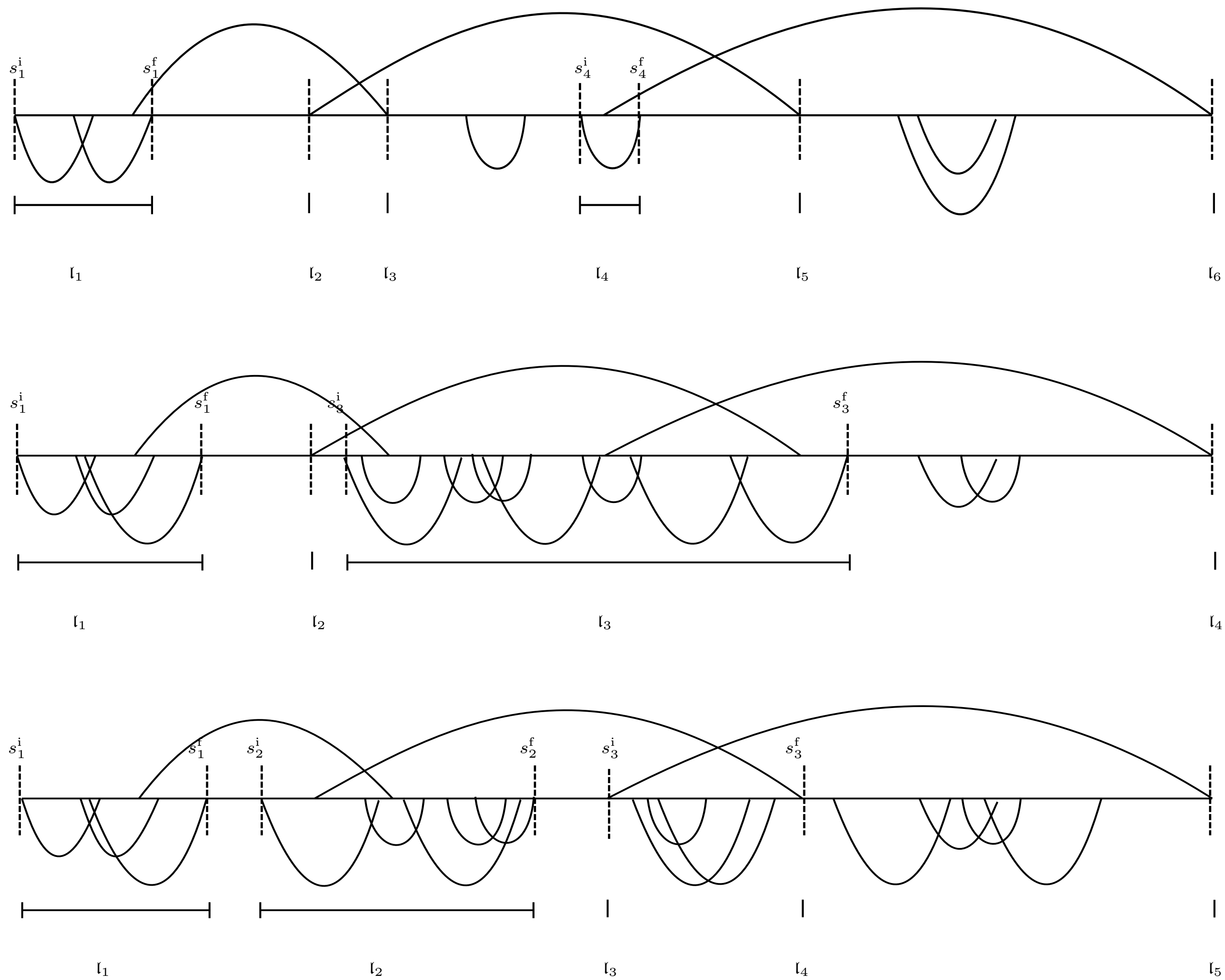


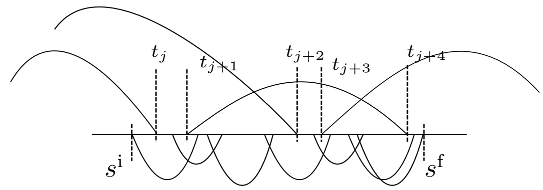
Figure 10: The picture shows three different choices of short diagrams $\sigma \in \Sigma_{[0,t]}(<\tau)$. Recall that short diagrams are drawn below the horizontal (time) axis. In each picture, we show the resulting vertex partition by listing the vertices $\mathfrak{l}_1, \mathfrak{l}_2, \ldots$. The dressed vertices are denoted by a horizontal bar whose endpoints represent the vertex time-coordinates $s_k^{\mathrm{i}}, s_k^{\mathrm{f}}$. The bare vertices are denoted by a short vertical line whose position represents the (dummy) vertex time coordinates $s_k^{\mathrm{i}} = s_k^{\mathrm{f}} = t_j$. The time-coordinates of the bare vertices are not shown since they coincide with time-coordinates of long pairings. For example, in the bottom picture, $\mathfrak{l}_1, \mathfrak{l}_2$ are dressed and $\mathfrak{l}_3, \mathfrak{l}_4, \mathfrak{l}_5$ are bare.

To write a formula for the vertex operator $\mathcal{B}(\mathfrak{l}_k, s_k^{\mathrm{i}}, s_k^{\mathrm{f}})$, we need to relabel the time-coordinates of σ_l and $\sigma \in \Sigma_{[s_k^{\mathrm{i}}, s_k^{\mathrm{f}}]}(<\tau, \mathrm{ir})$.

Consider the m triples $(t_i(\sigma_l), x_i(\sigma_l), l_i(\sigma_l))$, for $i = j, \ldots, j+m-1$, i.e., a subset of the $2|\sigma_l|$ triples determined by the long diagram σ_l, and the $2|\sigma|$ triples $t_i(\sigma), x_i(\sigma), l_i(\sigma)$ with $i = 1, \ldots, 2|\sigma|$ determined by $\sigma \in \Sigma_{[s_k^{\mathrm{i}}, s_k^{\mathrm{f}}]}(<\tau, \mathrm{ir})$. We now define the triples $(t_i'', x_i'', l_i'')_i^{m+2|\sigma|}$ by time-ordering (i.e. such that $t_i'' \leq t_{i+1}''$) of the union of triples

$$(t_i(\sigma), x_i(\sigma), l_i(\sigma))_{i=1}^{2|\sigma|} \quad \text{and} \quad (t_i(\sigma_l), x_i(\sigma_l), l_i(\sigma_l))_{i=j}^{j+m-1}. \tag{7.41}$$

$$\mathcal{B}(\mathfrak{l}, s^{\mathrm{i}}, s^{\mathrm{f}}) = \int_{\Sigma_{[s^{\mathrm{i}}, s^{\mathrm{f}}]}(<\tau, \mathrm{ir})} \mathrm{d}\sigma$$



Short diagrams σ

Figure 11: A part of a long diagram $\sigma_l \in \Sigma_{[0,t]}(>\tau)$ is shown, suggesting a dressed vertex $\mathfrak{l}$ with vertex set $S(\mathfrak{l}) = \{t_j, \ldots, t_{j+4}\}$. The end points of the pairings that are 'floating' in the air are immaterial to this vertex, as long as they land on the time-axis outside the interval $[s^{\mathrm{i}}, s^{\mathrm{f}}]$. The vertex operator $\mathcal{B}(\mathfrak{l}, s^{\mathrm{i}}, s^{\mathrm{f}})$ is obtained by integrating all diagrams in $\Sigma_{[s^i, s^f]}(<\tau, \mathrm{ir})$.

The vertex operator $\mathcal{B}(\mathfrak{l}_k, s^{\mathrm{i}}_k, s^{\mathrm{f}}_k)$ is then defined as follows

$$\mathcal{B}(\mathfrak{l}_k, s^{\mathrm{i}}_k, s^{\mathrm{f}}_k) := \int_{\Sigma_{[s^{\mathrm{i}}_k, s^{\mathrm{f}}_k]}(<\tau, \mathrm{ir})} \mathrm{d}\sigma\, \zeta(\sigma) \mathcal{V}_{[s^{\mathrm{i}}_k, s^{\mathrm{f}}_k]} \left((t''_i, x''_i, l''_i)_{i=1}^{m+2|\sigma|} \right) \tag{7.42}$$

where the dependence of the integrand on σ is implicit in the above definition of the triples (t''_i, x''_i, l''_i). The double primes in the coordinates (t''_i, x''_i, l''_i) are supposed to render the comparison with later formulas easier.

We now treat the simple case in which the vertex $\mathfrak{l}_k$ is bare. In that case, there is a j such that $S(\mathfrak{l}_k) = \{t_j\}$ and the vertex operator is simply defined as

$$\mathcal{B}(\mathfrak{l}_k, s^{\mathrm{i}}_k, s^{\mathrm{f}}_k) := \mathcal{I}_{x_j, l_j}, \qquad s^{\mathrm{i}}_k = s^{\mathrm{f}}_k = t_j \tag{7.43}$$

Hence, in this case, the vertex time-coordinates $s^{\mathrm{i}}_k, s^{\mathrm{f}}_k$ are dummy coordinates, see also Figure 10.

7.3.4 The operator $\mathcal{C}_t(\sigma_l, \mathfrak{L})$ as an integral over time-coordinates of vertex operators

We are ready to give a constructive formula for $\mathcal{C}_t(\sigma_l, \mathfrak{L})$, as announced in Section 7.3.1. First, we define the integration measure over the vertex time-coordinates $s^{\mathrm{i}}_k, s^{\mathrm{f}}_k$;

$$\mathcal{D}\underline{s}^{\mathrm{i}} \mathcal{D}\underline{s}^{\mathrm{f}} := \prod_{\substack{k = 1, \ldots, |\mathfrak{L}| \\ \mathfrak{l}_k \text{ dressed}}} \mathrm{d}s^{\mathrm{i}}_k \mathrm{d}s^{\mathrm{f}}_k \quad \left\{ \begin{array}{ll} \delta(s^{\mathrm{i}}_1) & \mathfrak{l}_1 \text{ dressed} \\ 1 & \mathfrak{l}_1 \text{ bare} \end{array} \right\} \times \left\{ \begin{array}{ll} \delta(s^{\mathrm{f}}_{|\mathfrak{L}|} - t) & \mathfrak{l}_{|\mathfrak{L}|} \text{ dressed} \\ 1 & \mathfrak{l}_{|\mathfrak{L}|} \text{ bare} \end{array} \right\} \tag{7.44}$$

To understand this formula, we observe that only non-dummy vertex time coordinates need to be integrated over. A dummy vertex time coordinate is a time coordinate whose value is a-priori fixed by σ_l and the vertex partition $\mathfrak{L}$. The non-dummy times are the time coordinates of the dressed vertices, except at the temporal boundaries $0, t$, where such a time coordinate is also a dummy coordinate. The terms between $\{\cdot\}$-brackets in formula (7.44) take care of this . Finally, the formula for $\mathcal{C}_t(\sigma_l, \mathfrak{L})$ is

$$\mathcal{C}_t(\sigma_l, \mathfrak{L}) = \int_{\substack{0 < s^{\mathrm{i}}_k < s^{\mathrm{f}}_k < t \\ s^{\mathrm{f}}_k < s^{\mathrm{i}}_{k'} \text{ for } k' > k}} \mathcal{D}\underline{s}^{\mathrm{i}} \mathcal{D}\underline{s}^{\mathrm{f}}\ \mathcal{B}(\mathfrak{l}_{|\mathfrak{L}|}, s^{\mathrm{i}}_{|\mathfrak{L}|}, s^{\mathrm{f}}_{|\mathfrak{L}|}) \mathcal{Z}^{\tau}_{s^{\mathrm{i}}_{|\mathfrak{L}|} - s^{\mathrm{f}}_{|\mathfrak{L}|-1}} \ldots \mathcal{Z}^{\tau}_{s^{\mathrm{i}}_3 - s^{\mathrm{f}}_2} \mathcal{B}(\mathfrak{l}_2, s^{\mathrm{i}}_2, s^{\mathrm{f}}_2) \mathcal{Z}^{\tau}_{s^{\mathrm{i}}_2 - s^{\mathrm{f}}_1} \mathcal{B}(\mathfrak{l}_1, s^{\mathrm{i}}_1, s^{\mathrm{f}}_1) \tag{7.45}$$

where the indices k, k' correspond to vertices $\mathfrak{l}_k, \mathfrak{l}_{k'}$: only the time-coordinates of dressed vertices are integrated over, even though all vertices appear on the RHS. This formula can be checked from the definition (7.37) and the explicit expressions for the vertex operators $\mathcal{B}(\cdot; \cdot, \cdot)$ above. The cutoff reduced dynamics $\mathcal{Z}^{\tau}_t$ in (7.45) appears by summing the small diagrams between the vertices, using formula (7.18).

8 The sum over "small" diagrams

In this section, we establish two results. First, we analyze the cutoff-dynamics $\mathcal{Z}_t^\tau$. The main bound is stated in Lemma 8.3, and a proof of Lemma 6.1 (concerning the Laplace transform of $\mathcal{Z}_t^\tau$) is outlined immediately after Lemma 8.3. Second, we resum the small subdiagrams within a general irreducible diagram: Recall that the conditional cutoff dynamics $\mathcal{C}_t(\sigma_l)$ is defined as the sum over all irreducible diagrams in $[0,t]$ containing the long diagram σ_l. In Lemma 8.6, we obtain a description of $\mathcal{C}_t(\sigma_l)$ that does not involve any small diagrams. In this sense, we have performed a *blocking procedure*, getting rid of information on time-scales smaller than τ.

Since this section uses parameters and constants that were introduced earlier in the paper, we encourage the reader to consult the overview tables in Section 9.4.

8.1 Generic constants

In Sections 8 and 9, we will state bounds that will depend in a crucial way on the parameters λ, γ and τ. The parameter γ is a momentum-like variable used to bound matrix elements in position representation, see below in Section 8.3. It appeared first in Section 4.3.1. To simplify the presentation, we introduce the following notation and conventions.

- We write $c(\gamma)$ for functions of $\gamma \geq 0$ with the property that $c(\gamma)$ is decreasing as $\gamma \nearrow \infty$, and $c(\gamma)$ is finite, except, possibly, at $\gamma = 0$. It is understood that $c(\gamma)$ is independent of λ.
- We write $c(\gamma,\lambda)$ for functions of $\gamma \geq 0$ and $\lambda \in \mathbb{R}$ that have the asymptotics

$$c(\gamma,\lambda) = o(\gamma^0)O(\lambda^2) + o(\lambda^2), \qquad \gamma \to 0,\ \lambda \to 0 \tag{8.1}$$

- We write $c'(\gamma,\lambda)$ for functions of $\gamma \geq 0$ and $\lambda \in \mathbb{R}$ that have the asymptotics

$$c'(\gamma,\lambda) = o(\gamma^0)O(\lambda^2) + c(\gamma)o(\lambda^2), \qquad \gamma \to 0,\ \lambda \to 0 \tag{8.2}$$

- The cutoff time $\tau = \tau(\lambda)$ is treated as an implicit function of λ, satisfying (5.13). In particular, $c(\gamma,\lambda)$ and $c'(\gamma,\lambda)$ can depend on τ.

8.2 Bounds in the sense of matrix elements

In Section 2.5 , we introduced the kernel notation $\mathcal{A}_{x_L,x_R;x'_L,x'_R}$, for operators $\mathcal{A}$ on $\mathcal{B}_2(l^2(\mathbb{Z}^d,\mathscr{S}))$; $\mathcal{A}_{x_L,x_R;x'_L,x'_R}$ is an element of $\mathscr{B}(\mathscr{B}_2(\mathscr{S}))$ such that

$$\langle S, \mathcal{A}S'\rangle = \sum_{x_L,x_R;x'_L,x'_R} \langle S(x_L,x_R), \mathcal{A}_{x_L,x_R;x'_L,x'_R} S'(x'_L,x'_R)\rangle_{\mathscr{B}_2(\mathscr{S})} \tag{8.3}$$

First, we introduce a notion that allows us to bound operators $\mathcal{A}$ by their 'matrix elements' $\mathcal{A}_{x_L,x_R;x'_L,x'_R}$.

Definition 8.1. Let $\mathcal{A}$ and $\tilde{\mathcal{A}}$ be operators on $\mathcal{B}_2(l^2(\mathbb{Z}^d,\mathscr{S}))$ and $\mathcal{B}_2(l^2(\mathbb{Z}^d))$, respectively. We say that $\tilde{\mathcal{A}}$ dominates $\mathcal{A}$ in 'the sense of matrix elements', denoted by

$$\mathcal{A} \underset{m.e.}{\leq} \tilde{\mathcal{A}}, \tag{8.4}$$

iff

$$\|\mathcal{A}_{x_L,x_R;x'_L,x'_R}\|_{\mathscr{B}(\mathscr{B}_2(\mathscr{S}))} \leq \tilde{\mathcal{A}}_{x_L,x_R;x'_L,x'_R} \tag{8.5}$$

Note that, if $\mathcal{A}$ is an operator on $\mathcal{B}_2(l^2(\mathbb{Z}^d))$, the inequality $\mathcal{A} \underset{m.e.}{\leq} \tilde{\mathcal{A}}$ literally means that the absolute values of the matrix elements of $\mathcal{A}$ are smaller than the matrix elements of $\tilde{\mathcal{A}}$. We will need the following implication

$$\mathcal{A} \underset{m.e.}{\leq} \tilde{\mathcal{A}} \qquad \Rightarrow \qquad \|\mathcal{A}\| \leq \|\tilde{\mathcal{A}}\| \tag{8.6}$$

Indeed, for any $S \in \mathscr{B}_2(l^2(\mathbb{Z}^d) \otimes \mathscr{S}) \sim l^2(\mathbb{Z}^d \times \mathbb{Z}^d, \mathscr{B}_2(\mathscr{S}))$, we construct

$$\tilde{S}(x_L, x_R) := \|S(x_L, x_R)\|_{\mathscr{B}_2(\mathscr{S})} \tag{8.7}$$

such that $\|\tilde{S}\|_{l^2(\mathbb{Z}^d \times \mathbb{Z}^d)} = \|S\|_{l^2(\mathbb{Z}^d \times \mathbb{Z}^d, \mathscr{B}_2(\mathscr{S}))}$ and

$$|\langle S, \mathcal{A}S' \rangle| \leq \langle \tilde{S}, \tilde{\mathcal{A}}\tilde{S}' \rangle \tag{8.8}$$

from which (8.6) follows.

8.3 Bounding operators

We introduce operators on $\mathscr{B}_2(l^2(\mathbb{Z}^d))$ that will be used as upper bounds 'in the sense of matrix elements', as defined above. These bounding operators will depend on the coupling constant λ, the conjugation parameter $\gamma > 0$ and the cutoff time $\tau = \tau(\lambda)$. Let the function $r_\tau(\gamma, \lambda)$ and the constants $c^1_{\mathcal{Z}}, c^2_{\mathcal{Z}}$ be as defined in Lemma 6.2 and, in addition, let

$$r_\varepsilon(\gamma, \lambda) := 2\lambda^2 q_\varepsilon(2\gamma), \qquad \text{for } 2\gamma \leq \delta_\varepsilon, \tag{8.9}$$

with $q_\varepsilon(\cdot)$ and δ_ε as in Assumption 2.1. We define

$$(\tilde{\mathcal{I}}_{x,l})_{x_L,x_R;x'_L,x'_R} := \delta_{x_L,x'_L}\delta_{x_R,x'_R}\left(\delta_{l=L}\delta_{x_L=x} + \delta_{l=R}\delta_{x_R=x}\right) \tag{8.10}$$

$$\begin{aligned}(\tilde{\mathcal{Z}}^{\tau,\gamma}_t)_{x_L,x_R;x'_L,x'_R} &:= c^1_{\mathcal{Z}}\, \mathrm{e}^{r_\tau(\gamma,\lambda)t}\mathrm{e}^{-\frac{\gamma}{2}\left|(x'_L+x'_R)-(x_L+x_R)\right|}\mathrm{e}^{-\gamma|x_L-x_R|}\mathrm{e}^{-\gamma\left|x'_L-x'_R\right|} \\ &+ c^2_{\mathcal{Z}}\, \mathrm{e}^{-\lambda^2 g_c t}\mathrm{e}^{-\frac{\gamma}{2}\left|(x'_L+x'_R)-(x_L+x_R)\right|}\mathrm{e}^{-\gamma\left|(x_L-x_R)-(x'_L-x'_R)\right|}\end{aligned} \tag{8.11}$$

$$(\tilde{\mathcal{U}}^{\gamma}_t)_{x_L,x_R;x'_L,x'_R} := \mathrm{e}^{r_\varepsilon(\gamma,\lambda)t}\mathrm{e}^{-\frac{\gamma}{2}\left|(x'_L+x'_R)-(x_L+x_R)\right|}\mathrm{e}^{-\gamma\left|(x_L-x_R)-(x'_L-x'_R)\right|} \tag{8.12}$$

In order for definitions (8.11, 8.12) to make sense, λ and $\gamma > 0$ have to be sufficiently small, such that the functions $r_\varepsilon(\gamma, \lambda)$ and $r_\tau(\gamma, \lambda)$ are well-defined. In particular, we need conditions on λ and γ such that Lemma 6.2 applies.

The operators $\tilde{\mathcal{I}}_{z,l}, \tilde{\mathcal{Z}}^{\tau,\gamma}_t, \tilde{\mathcal{U}}^{\gamma}_t$ inherit their notation from the operators they are designed to bound, as we have the following inequalities, for λ, γ small enough:

$$\mathcal{I}_{x,l} \underset{m.e.}{\leq} \tilde{\mathcal{I}}_{x,l} \tag{8.13}$$

$$\mathcal{Z}^{\tau}_t \underset{m.e.}{\leq} \tilde{\mathcal{Z}}^{\tau,\gamma}_t \tag{8.14}$$

$$\mathcal{U}_t \underset{m.e.}{\leq} \tilde{\mathcal{U}}^{\gamma}_t \tag{8.15}$$

The first inequality is obvious from the definition of $\mathcal{I}_{x,l}$ in (5.7) and the fact that $\|W\|_{\mathscr{B}(\mathscr{S})} \leq 1$. Indeed, $\tilde{\mathcal{I}}_{x,l}$ can be obtained from $\mathcal{I}_{x,l}$ by replacing W by 1. The second inequality is the result of Lemma 6.2 and the third inequality follows from the bounds following Assumption 2.1.

We start by stating obvious rules to multiply the operators $\tilde{\mathcal{Z}}^{\tau,\gamma}_t$ and $\tilde{\mathcal{U}}^{\gamma}_t$.

Lemma 8.1. *For λ, γ small enough, the following bounds hold (with $c(\gamma)$ and $c(\gamma, \lambda)$ as defined in Section 8.1)*

- *For all sequences of times $s_1, \ldots, s_n$ with $t = \sum_{i=1}^n s_i$,*

$$\tilde{\mathcal{U}}^{\gamma}_{s_n} \ldots \tilde{\mathcal{U}}^{\gamma}_{s_2}\tilde{\mathcal{U}}^{\gamma}_{s_1} \underset{m.e.}{\leq} [c(\gamma)]^{n-1}\, \mathrm{e}^{c(\gamma,\lambda)t}\, \tilde{\mathcal{U}}^{\frac{\gamma}{2}}_t \tag{8.16}$$

$$\tilde{\mathcal{Z}}^{\tau,\gamma}_{s_n} \ldots \tilde{\mathcal{Z}}^{\tau,\gamma}_{s_2}\tilde{\mathcal{Z}}^{\tau,\gamma}_{s_1} \underset{m.e.}{\leq} [c(\gamma)]^{n-1}\, \mathrm{e}^{c(\gamma,\lambda)t}\, \tilde{\mathcal{Z}}^{\tau,\frac{\gamma}{2}}_t \tag{8.17}$$

- *For all times $s < t$,*

$$\tilde{\mathcal{Z}}^{\tau,\gamma}_{t-s}\tilde{\mathcal{U}}^{\gamma}_{s} \underset{m.e.}{\leq} c(\gamma)\mathrm{e}^{\frac{s}{2\tau}}\mathrm{e}^{c(\gamma,\lambda)t}\tilde{\mathcal{Z}}^{\tau,\frac{\gamma}{2}}_{t} \tag{8.18}$$

$$\tilde{\mathcal{U}}^{\gamma}_{t-s}\tilde{\mathcal{Z}}^{\tau,\gamma}_{s} \underset{m.e.}{\leq} c(\gamma)\mathrm{e}^{\frac{t-s}{2\tau}}\mathrm{e}^{c(\gamma,\lambda)t}\tilde{\mathcal{Z}}^{\tau,\frac{\gamma}{2}}_{t} \tag{8.19}$$

Proof. Inequalities (8.16) and (8.17) are immediate consequences of the fact that

$$\sum_{x\in\mathbb{Z}^d}\mathrm{e}^{-\gamma|x-x_1|}\mathrm{e}^{-\gamma|x-x_2|} \leq e^{-\frac{\gamma}{2}|x_1-x_2|}\sum_{x\in\mathbb{Z}^d}\mathrm{e}^{-\frac{\gamma}{2}|x|}, \qquad \text{for any } \gamma > 0 \tag{8.20}$$

To derive inequalities (8.18) and (8.19), we use (8.20) and we dominate exponential factors $\mathrm{e}^{O(\lambda^2)t}$ on the RHS by $\mathrm{e}^{\frac{t}{2\tau}}$, using that $\tau\lambda^2 \to 0$ as $\lambda \searrow 0$.

□

Lemma 8.2, below, shows how the bounds of Lemma 8.1 are used to integrate over diagrams. This lemma will be used repeatedly in the next sections, and, since it is a crucial step, we treat the following simple example in detail: We attempt to bound the expression

$$\mathcal{F} := \int\limits_{s^{\mathrm{i}}<t_1<t_2<s^{\mathrm{f}}} \mathrm{d}t_1\mathrm{d}t_2 \sum_{x_1,x_2,l_1,l_2} \zeta(\sigma)\underbrace{\tilde{\mathcal{U}}^{\gamma}_{s^{\mathrm{f}}-t_2}\tilde{\mathcal{I}}_{x_2,l_2}\tilde{\mathcal{U}}^{\gamma}_{t_2-t_1}\tilde{\mathcal{I}}_{x_1,l_1}\tilde{\mathcal{U}}^{\gamma}_{t_1-s^{\mathrm{i}}}}_{=:\tilde{\mathcal{A}}} \tag{8.21}$$

in "the sense of matrix elements", with σ being the diagram in $\Sigma^1_{[s^{\mathrm{i}},s^{\mathrm{f}}]}$ consisting of the ordered pair $((t_1,x_1,l_1),(t_2,x_2,l_2))$. We proceed as follows:

1) We bound $\zeta(\sigma)$ by $\sup_{x_1,x_2,l_1,l_2}|\zeta(\sigma)|$. Note that the latter expression is a function of t_2-t_1 only.
2) Since the only dependence on x_1,x_2,l_1,l_2 is in the operators $\tilde{\mathcal{I}}_{x_i,l_i}$, we perform the sum $\sum_{x_i,l_i}\tilde{\mathcal{I}}_{x_i,l_i} = 1$, for $i=1,2$.
3) Since the operators $\tilde{\mathcal{I}}_{x_i,l_i}$ have disappeared, we can bound

$$\tilde{\mathcal{U}}^{\gamma}_{s^{\mathrm{f}}-t_2}\tilde{\mathcal{U}}^{\gamma}_{t_2-t_1}\tilde{\mathcal{U}}^{\gamma}_{t_1-s^{\mathrm{i}}} \underset{m.e.}{\leq} [c(\gamma)]^2\mathrm{e}^{c(\gamma,\lambda)|s^{\mathrm{f}}-s^{\mathrm{i}}|}\ \tilde{\mathcal{U}}^{\frac{\gamma}{2}}_{s^{\mathrm{f}}-s^{\mathrm{i}}} \tag{8.22}$$

   using Lemma 8.1.

Thus

$$\mathcal{F} \underset{m.e.}{\leq} \tilde{\mathcal{U}}^{\frac{\gamma}{2}}_{s^{\mathrm{f}}-s^{\mathrm{i}}}\,\mathrm{e}^{c(\gamma,\lambda)|s^{\mathrm{f}}-s^{\mathrm{i}}|}\int\limits_{s^{\mathrm{i}}<t_1<t_2<s^{\mathrm{f}}}\mathrm{d}t_1\mathrm{d}t_2\,[c(\gamma)]^2\sup_{x_1,x_2,l_1,l_2}|\zeta(\sigma)| \tag{8.23}$$

Note that $\sup_{x_1,x_2,l_1,l_2}|\zeta(\sigma)| = \sup_x|\psi(x,t_2-t_1)|$ because $|\sigma|=1$. The short derivation above can be considered to be an application of Lemma 7.1, as we illustrate by writing

$$\mathcal{F}_{x_L,x_R;x'_L,x'_R} = \int\limits_{\Sigma^1_{[s^{\mathrm{i}},s^{\mathrm{f}}]}}\mathrm{d}\sigma G(\sigma)F(\sigma), \qquad \text{with } G(\sigma)=\zeta(\sigma),\ F(\sigma):=\tilde{\mathcal{A}}_{x_L,x_R;x'_L,x'_R} \tag{8.24}$$

and hence (8.23) follows from Lemma 7.1 after applying (8.22).

Lemma 8.2 is a generalization of the bound (8.23) above.

Lemma 8.2. *Fix an interval $I=[s^{\mathrm{i}},s^{\mathrm{f}}]$ and a set of m triples $(t'_i,x'_i,l'_i)_{i=1}^m$ such that $t'_i\in I$ and $t'_i<t'_{i+1}$. For any $\sigma\in\Sigma_I(\mathrm{ir})$, we define the set of $n:=m+2|\sigma|$ triples $(t''_i,x''_i,l''_i)_{i=1}^{m+2|\sigma|}$ by time-ordering (i.e., such that $t''_i\leq t''_{i+1}$) of the union of triples*

$$(t'_i,x'_i,l'_i)_{i=1}^m, \qquad \text{and} \qquad (t_i(\sigma),x_i(\sigma),l_i(\sigma))_{i=1}^{2|\sigma|} \tag{8.25}$$

Then

$$\int_{\Sigma_I(\mathrm{ir})} \mathrm{d}\sigma\, |\zeta(\sigma)|\; \tilde{\mathcal{I}}_{x''_n,l''_n} \tilde{\mathcal{U}}^{\gamma}_{t''_n - t''_{n-1}} \ldots \tilde{\mathcal{U}}^{\gamma}_{t''_3 - t''_2} \tilde{\mathcal{I}}_{x''_2,l''_2} \tilde{\mathcal{U}}^{\gamma}_{t''_2 - t''_1} \tilde{\mathcal{I}}_{x''_1,l''_1} \tag{8.26}$$

$$\underset{m.e.}{\leq} \left(\mathrm{e}^{c(\gamma,\lambda)|I|} \int_{\Pi_T \Sigma_I(\mathrm{ir})} \mathrm{d}[\sigma]\, [c(\gamma)]^{2|\sigma|} \sup_{\underline{x}(\sigma),\underline{l}(\sigma)} |\zeta(\sigma)| \right) \quad \times \quad \tilde{\mathcal{U}}^{\frac{\gamma}{2}}_{s^{\mathrm{f}} - t'_m} \tilde{\mathcal{I}}_{x'_m,l'_m} \tilde{\mathcal{U}}^{\frac{\gamma}{2}}_{t'_m - t'_{m-1}} \ldots \tilde{\mathcal{I}}_{x'_2,l'_2} \tilde{\mathcal{U}}^{\frac{\gamma}{2}}_{t'_2 - t'_1} \tilde{\mathcal{I}}_{x'_1,l'_1} \tilde{\mathcal{U}}^{\frac{\gamma}{2}}_{t'_1 - s^{\mathrm{i}}}$$

Moreover, the statement remain true if one replaces $\tilde{\mathcal{U}}^{\gamma}_t \to \tilde{\mathcal{Z}}^{\tau,\gamma}_t$ *on the LHS and* $\tilde{\mathcal{U}}^{\frac{\gamma}{2}}_t \to \tilde{\mathcal{Z}}^{\tau,\frac{\gamma}{2}}_t$ *on the RHS of* (8.26).

Proof. The proof is a copy of the proof of the the bound (8.23). The steps are

1) Dominate $|\zeta(\sigma)|$ by $\sup_{\underline{x}(\sigma),\underline{l}(\sigma)} |\zeta(\sigma)|$
2) Sum over $\underline{x}(\sigma), \underline{l}(\sigma)$ by using $\sum_{x_i,l_i} \tilde{\mathcal{I}}_{x_i,l_i} = 1$
3) Multiply the operators $\tilde{\mathcal{U}}^{\gamma}_t$ or $\tilde{\mathcal{Z}}^{\tau,\gamma}_t$, using the bound (8.16) or (8.17).
4) Interpret the remaining sum over $|\sigma|$ and integration over $\underline{t}(\sigma)$ as an integration over equivalence classes $[\sigma]$.

□

8.4 Bound on short pairings and proof of Lemma 6.1

We recall that the crucial result in Lemma 6.1(see Statement 2 therein) is the bound

$$\mathcal{J}_\kappa \mathcal{R}^{\tau}_{ex}(z) \mathcal{J}_{-\kappa} = \int_{\mathbb{R}^+} \mathrm{d}t\, \mathrm{e}^{-tz} \int_{\Sigma_{[0,t]}(<\tau,\mathrm{ir})} \mathrm{d}\sigma\, 1_{|\sigma|\geq 2} \mathcal{J}_\kappa \mathcal{V}_{[0,t]}(\sigma) \mathcal{J}_{-\kappa} = O(\lambda^2) O(\lambda^2 \tau), \tag{8.27}$$

uniformly for $\mathrm{Re}\, z \geq -\frac{1}{2\tau}$ and for $|\mathrm{Im}\, \kappa|$ small enough.

In the **first** step of the proof of (8.27), we sum over the $\underline{x}(\sigma), \underline{l}(\sigma)$- coordinates of the diagrams in (8.27). The strategy for doing this has been outlined in Sections 8.2 and 8.3.

Lemma 8.3. *For* λ, γ *smalll enough,*

$$\int_{\Sigma_{[0,t]}(<\tau,\mathrm{ir})} \mathrm{d}\sigma\, 1_{|\sigma|\geq 2} \zeta(\sigma) \mathcal{V}_{[0,t]}(\sigma) \quad \underset{m.e.}{\leq} \quad \mathrm{e}^{c(\gamma,\lambda)t}\, \tilde{\mathcal{U}}^{\gamma}_t \int_{\Pi_T \Sigma_{[0,t]}(<\tau,\mathrm{ir})} \mathrm{d}[\sigma]\, c(\gamma)^{|\sigma|}\, 1_{|\sigma|\geq 2} \sup_{\underline{x}(\sigma),\underline{l}(\sigma)} |\zeta(\sigma)| \tag{8.28}$$

Proof. In the definition of $\mathcal{V}_I(\sigma)$, see e.g. (7.16), we bound $\mathcal{I}_{x,l}$ by $\tilde{\mathcal{I}}_{x,l}$ and $\mathcal{U}_t$ by $\tilde{\mathcal{U}}^{\gamma}_t$. Then, we use the bound (8.26) with $m = 0$ to obtain (8.28). Note that, since $m = 0$, the set of triples $(t''_i, x''_i, l''_i)_{i=1}^{m+2|\sigma|}$ is equal to the set of triples $(t_i(\sigma), x_i(\sigma), l_i(\sigma))_{i=1}^{2|\sigma|}$. Note also that we use (8.26) with $\Sigma_I(<\tau, \mathrm{ir})$ instead of $\Sigma_I(\mathrm{ir})$, and with the restriction to $|\sigma| \geq 2$. However, this does not change the validity of (8.26), as one easily checks. □

To appreciate how Lemma 8.3 relates to the bound (8.27), we note already that the bound (8.28) remains true if one puts left and right hand sides between $\mathcal{J}_\kappa \cdot \mathcal{J}_{-\kappa}$ for purely imaginary κ (for general κ the matrix elements can become negative, which is not allowed by our definition of $\underset{m.e.}{\leq}$).

In the **second** step of the proof, we estimate the Laplace transform of the integral over equivalence classes $[\sigma]$ appearing on the RHS of (8.28). This estimate uses three important facts

1) The correlation functions in (8.28) decay exponentially with rate $1/\tau$, due to the cutoff.
2) The diagrams are restricted to $|\sigma| \geq 2$, they are therefore subleading with respect to a diagram with $|\sigma| = 1$.
3) We allow the estimate to depend on γ in a non-uniform way. Indeed, γ will be fixed in the last step of the argument.

Concretely, we show that, for $0 < a \leq \frac{1}{\tau}$ and for λ small enough (depending on γ)

$$\int_{\mathbb{R}^+} \mathrm{d}t\, \mathrm{e}^{at} \int_{\Sigma_{[0,t]}(<\tau,\mathrm{ir})} \mathrm{d}[\sigma]\, 1_{|\sigma|\geq 2} \left([c(\gamma)]^{2|\sigma|} \sup_{\underline{x}(\sigma),\underline{l}(\sigma)} |\zeta(\sigma)| \right) = O(\lambda^2) O(\lambda^2\tau) c(\gamma), \qquad \lambda \searrow 0, \lambda^2\tau \searrow 0 \tag{8.29}$$

To verify (8.29), we set

$$k(t) := \lambda^2 c(\gamma)\, 1_{|t|\leq\tau} \sup_x |\psi(x,t)| \tag{8.30}$$

and we calculate, by exploiting the cutoff τ in the definition of $k(\cdot)$,

$$\|\mathrm{e}^{\frac{1}{\tau}t} k\|_1 < \lambda^2 c(\gamma), \qquad \|t\mathrm{e}^{(\frac{1}{\tau}+\|k\|_1)t} k\|_1 = \tau O(\lambda^2) c(\gamma) \tag{8.31}$$

The norm $\|\cdot\|_1$ refers to the variable t, i.e., $\|h\|_1 = \int_0^\infty \mathrm{d}t |h(t)|$. Hence (8.29) follows from the bound (D.4) in Lemma D.1, in Appendix D, after using that $\tau O(\lambda^2) < C$, as $\lambda \searrow 0$, and choosing λ small enough.

In the **third** step of the proof, we fix γ. By using the explicit form (8.12) and the relation (2.60), we check that

$$\left\| \left(\mathcal{J}_\kappa \tilde{\mathcal{U}}_t^\gamma \mathcal{J}_{-\kappa} \right)_{x_L,x_R;x'_L,x'_R} \right\| \leq \mathrm{e}^{c(\gamma,\lambda)t}, \qquad \text{for any} \qquad |\mathrm{Im}\, \kappa_{L,R}| < \gamma. \tag{8.32}$$

Next, we make use of the following general fact that can be easily checked (e.g., by the Cauchy-Schwarz inequality): If, for some $\gamma > 0$ and $C < \infty$,

$$\| \left(\mathcal{J}_\kappa \mathcal{A} \mathcal{J}_{-\kappa} \right)_{x_L,x_R;x'_L,x'_R} \| \leq C, \qquad \text{uniformly for } \kappa_{L,R} \text{ s.t. } |\mathrm{Im}\, \kappa_{L,R}| \leq \gamma, \tag{8.33}$$

then

$$\|\mathcal{A}\| \leq c(\gamma) \tag{8.34}$$

where the norms refer to the operator norm on $\mathscr{B}(\mathscr{B}_2(l^2(\mathbb{Z}^d,\mathscr{S})))$, as in Definition 8.1.

Hence, from (8.32) we get

$$\sup_{|\mathrm{Im}\, \kappa_{L,R}|<\gamma/2} \|\mathcal{J}_{-\kappa} \tilde{\mathcal{U}}_t^\gamma \mathcal{J}_\kappa\| \leq c(\gamma) \mathrm{e}^{c(\gamma,\lambda)t}. \tag{8.35}$$

By the first equality in (8.27) and Lemma 8.3,

$$\mathcal{J}_\kappa \mathcal{R}_{ex}(z) \mathcal{J}_{-\kappa} \underset{m.e.}{\leq} \int_{\mathbb{R}^+} \mathrm{d}t\, \mathrm{e}^{-t\mathrm{Re}\, z} \mathcal{J}_\kappa \tilde{\mathcal{U}}_t^\gamma \mathcal{J}_{-\kappa} \mathrm{e}^{c(\gamma,\lambda)t} \int_{\Sigma_{[0,t]}(<\tau,\mathrm{ir})} \mathrm{d}[\sigma]\, 1_{|\sigma|\geq 2} [c(\gamma)]^{2|\sigma|} \sup_{\underline{x}(\sigma),\underline{l}(\sigma)} |\zeta(\sigma)| \tag{8.36}$$

We combine (8.36) and (8.35) with (8.29), setting

$$a \equiv \max\left(-\mathrm{Re}\, z, 0\right) + c(\gamma,\lambda) \tag{8.37}$$

for λ small enough such that $c(\gamma,\lambda) \leq \frac{1}{2\tau}$ and such that (8.29) applies. At this point, the parameter γ has been fixed and this choice determines the maximal value of $|\mathrm{Im}\, \kappa|$. This concludes the proof of the bound in (8.27). The other statements of Lemma 6.1 are proven below.

Proof of Lemma 6.1 The claim about $\mathcal{R}_{ld}^\tau(z)$ (Statement 2) follows by a drastically simplified version of the above argument for $\mathcal{R}_{ex}^\tau(z)$.

To establish the convergence claim in Statement 1) of Lemma 6.1, it suffices, by (7.27), to check that $\|\mathcal{Z}_t^{\tau,\mathrm{ir}}\| \leq \mathrm{e}^{Ct}$ for some constant C. This has been established in the proof of Statement 2), above, since $\mathcal{R}_{ld}^\tau(z) + \mathcal{R}_{ex}^\tau(z)$ is the Laplace transform of $\mathcal{Z}_t^{\tau,\mathrm{ir}}$. The identity (6.4) was established in Section 7.2.

To check Statement 3), we employ expression (C.1) for $\mathcal{L}(z)$ and (7.28) for $\mathcal{R}^{\tau}_{ld}(z)$. The latter differs from $\mathcal{L}(z)$ in that it is the Laplace transform of a quanitity with a cutoff at $t = \tau$ and in the fact that it includes the propagator $\mathcal{U}_t = \mathrm{e}^{\mathrm{i}(\mathrm{ad}(Y)+\lambda^2 \mathrm{ad}(\varepsilon))t}$ whereas $\mathcal{L}(z)$ includes only $\mathrm{e}^{\mathrm{i}(\mathrm{ad}(Y))t}$. We observe that

$$\left\| \mathcal{J}_\kappa \left(\mathcal{R}^{\tau}_{\mathrm{ld}}(z) - \lambda^2 \mathcal{L}(z) \right) \mathcal{J}_{-\kappa} \right\| \leq 4\lambda^2 \int_\tau^\infty \mathrm{d}t \sup_x |\psi(x,t)| \left\| \mathcal{J}_\kappa \mathrm{e}^{-\mathrm{iad}(Y)t} \mathcal{J}_{-\kappa} \right\| \tag{8.38}$$

$$+ \quad 4\lambda^2 \int_0^\tau \mathrm{d}t \sup_x |\psi(x,t)| \left\| \mathcal{J}_\kappa \mathrm{e}^{-\mathrm{iad}(Y)t} \left(\mathrm{e}^{\mathrm{i}\lambda^2 \mathrm{ad}(\varepsilon(P))t} - 1 \right) \mathcal{J}_{-\kappa} \right\| \tag{8.39}$$

where the factors '4' originate from the sum over l_1, l_2 and we use that $\mathrm{Re}\, z \geq 0$. In the first term on the RHS, $\left\| \mathcal{J}_\kappa \mathrm{e}^{-\mathrm{iad}(Y)t} \mathcal{J}_{-\kappa} \right\| = 1$ since Y commutes with the position operator X. The second term is bounded by

$$4\lambda^2 \int_0^\tau \mathrm{d}s \sup_x |\psi(x,s)| \times \sup_{t \leq \tau} \left(\lambda^2 t \left\| \mathcal{J}_\kappa \mathrm{ad}(\varepsilon(P)) \mathrm{e}^{\mathrm{i}\lambda^2 \mathrm{ad}(\varepsilon(P))t} \mathcal{J}_{-\kappa} \right\| \right) \leq \tau \lambda^4 C \tag{8.40}$$

where we have used Lemma 5.2 and the bound (2.12). □

8.5 Bound on the vertex operators $\mathcal{B}(\mathfrak{l}, s^{\mathrm{i}}, s^{\mathrm{f}})$

In this section, we prove a bound on the 'dressed vertex operators', which were introduced in Section 7.3.3. Since such 'dressed vertex operators' contain an irreducible short diagram in the interval $[s^{\mathrm{i}}, s^{\mathrm{f}}]$, we obtain a bound that is exponentially decaying in $|s^{\mathrm{f}} - s^{\mathrm{i}}|$. In (8.41), this exponential decay resides in the function $w(\cdot)$ and it is made explicit through the calculation in (8.45).

The proof of the next lemma parallels the proof of Lemma 8.3 above. Consider m triples $(t'_i, x'_i, l'_i)_{i=1}^m$ and let $\mathfrak{l}$ be a (dressed) vertex with vertex set $S(\mathfrak{l}) = \{t'_1, \ldots, t'_m\}$. Let $s^{\mathrm{i}}, s^{\mathrm{f}}$ be vertex time-coordinates associated to $\mathfrak{l}$, i.e., such that $s^{\mathrm{i}} < t'_1$ and $s^{\mathrm{f}} > t'_m$.

Lemma 8.4. *For λ, γ small enough, the following bound holds:*

$$\mathcal{B}(\mathfrak{l}, s^{\mathrm{i}}, s^{\mathrm{f}}) \underset{m.e.}{\leq} w(s^{\mathrm{f}} - s^{\mathrm{i}}) \tilde{\mathcal{U}}^{\gamma}_{s^{\mathrm{f}} - t'_m} \tilde{\mathcal{I}}_{x'_m, l'_m} \tilde{\mathcal{U}}^{\gamma}_{t'_m - t'_{m-1}} \ldots \tilde{\mathcal{U}}^{\gamma}_{t'_2 - t'_1} \tilde{\mathcal{I}}_{x'_1, l'_1} \tilde{\mathcal{U}}^{\gamma}_{t'_1 - s^{\mathrm{i}}} \tag{8.41}$$

where

$$w(s^{\mathrm{f}} - s^{\mathrm{i}}) := \mathrm{e}^{\lambda^2 C' |s^{\mathrm{f}} - s^{\mathrm{i}}|} \int_{\Pi_T \Sigma_{[s^{\mathrm{i}}, s^{\mathrm{f}}]}(<\tau, \mathrm{ir})} \mathrm{d}[\sigma]\, C^{|\sigma|} \sup_{\underline{x}(\sigma), \underline{l}(\sigma)} |\zeta(\sigma)| \tag{8.42}$$

The RHS of (8.42) *indeed depends only on $s^{\mathrm{f}} - s^{\mathrm{i}}$, since the correlation function $\zeta(\sigma)$ depends only on differences of the time-coordinates of σ. The function $w(\cdot)$ depends on the coupling strength λ via the correlation function $\zeta(\sigma)$, see* (7.15)*.*

Proof. Starting from the definition of the vertex operator $\mathcal{B}(\mathfrak{l}, s^{\mathrm{i}}, s^{\mathrm{f}})$ given in (7.42), we bound the operators $\mathcal{I}_{x,l}, \mathcal{U}_t$ by $\tilde{\mathcal{I}}_{x,l}, \tilde{\mathcal{U}}^{\gamma}_t$ and we apply Lemma 8.2 to obtain

$$\begin{aligned} \mathcal{B}(\mathfrak{l}, s^{\mathrm{i}}, s^{\mathrm{f}}) \underset{m.e.}{\leq} \mathrm{e}^{c(\gamma,\lambda)|s^{\mathrm{f}} - s^{\mathrm{i}}|} & \int_{\Pi_T \Sigma_{[s^{\mathrm{i}}, s^{\mathrm{f}}]}(<\tau, \mathrm{ir})} \mathrm{d}[\sigma]\, [c(\gamma)]^{2|\sigma|} \sup_{\underline{x}(\sigma), \underline{l}(\sigma)} |\zeta(\sigma)| \\ & \tilde{\mathcal{U}}^{\frac{\gamma}{2}}_{s^{\mathrm{f}} - t'_m} \tilde{\mathcal{I}}_{x'_m, l'_m} \tilde{\mathcal{U}}^{\frac{\gamma}{2}}_{t'_m - t'_{m-1}} \ldots \tilde{\mathcal{U}}^{\frac{\gamma}{2}}_{t'_2 - t'_1} \tilde{\mathcal{I}}_{x'_1, l'_1} \tilde{\mathcal{U}}^{\frac{\gamma}{2}}_{t'_1 - s^{\mathrm{i}}} \end{aligned} \tag{8.43}$$

From the definition of $\tilde{\mathcal{U}}^{\gamma}_t$ in (8.12), we see that

$$\tilde{\mathcal{U}}^{\gamma_1}_t \leq \mathrm{e}^{c(\gamma,\lambda)t} \tilde{\mathcal{U}}^{\gamma_2}_t, \qquad \text{for } \gamma_2 < \gamma_1 \tag{8.44}$$

We dominate the RHS of (8.43) by fixing $\gamma/2 = \gamma_1$ and applying (8.44) for any $\gamma_2 \leq \gamma_1$. This yields (8.41), with the constant C in (8.42) given by fixing $\gamma = \gamma_1$ in $c(\gamma)$. One sees that the maximal value we can choose for γ_1 is $\gamma_1 = \frac{1}{4}\delta_\varepsilon$, with δ_ε as in Assumption 2.1. □

For later use, we note here that, for λ sufficiently small and with $w(t)$ defined in (8.42);

$$\begin{aligned}\int_{\mathbb{R}^+} \mathrm{d}t\,|t| w(t) \mathrm{e}^{\frac{t}{2\tau}} &\leq \tau C \int_{\mathbb{R}^+} \mathrm{d}t w(t) \mathrm{e}^{\frac{t}{\tau}} \\ &\leq O(\lambda^2 \tau^2), \qquad \lambda \searrow 0, \lambda^2\tau \searrow 0\end{aligned} \tag{8.45}$$

where the second inequality follows by the bound (D.3) in Lemma D.1, with

$$k(t) := \lambda^2 C \sup_x |\psi(x,t)| 1_{t \leq \tau} \qquad \text{and} \qquad a := \frac{1}{\tau} \tag{8.46}$$

for λ such that $\lambda^2 C' < 1/\tau$ with C' as in the exponent of (8.42).

8.6 Bound on the conditional cutoff dynamics $\mathcal{C}_t(\sigma_l)$

In this section, we state bounds on $\mathcal{C}_t(\sigma_l, \mathfrak{L})$ and $\mathcal{C}_t(\sigma_l)$, defined in Sections 7.2.1 and 7.3.1, respectively. Our bounds will follow in a straightforward way from Lemma 8.4 and formula (7.45), which we repeat here for convenience

$$\mathcal{C}_t(\sigma_l, \mathfrak{L}) = \int\limits_{\substack{0 < s^{\mathrm{i}}_k < s^{\mathrm{f}}_k < t \\ s^{\mathrm{f}}_k < s^{\mathrm{i}}_{k'} \text{ for } k' > k}} \mathcal{D}\underline{s}^{\mathrm{i}} \mathcal{D}\underline{s}^{\mathrm{f}}\ \mathcal{B}(\mathfrak{l}_{|\mathfrak{L}|}, s^{\mathrm{i}}_{|\mathfrak{L}|}, s^{\mathrm{f}}_{|\mathfrak{L}|}) \mathcal{Z}^{\tau}_{s^{\mathrm{f}}_{|\mathfrak{L}|} - s^{\mathrm{i}}_{|\mathfrak{L}|-1}} \dots \mathcal{Z}^{\tau}_{s^{\mathrm{f}}_3 - s^{\mathrm{i}}_2} \mathcal{B}(\mathfrak{l}_2, s^{\mathrm{i}}_2, s^{\mathrm{f}}_2) \mathcal{Z}^{\tau}_{s^{\mathrm{f}}_2 - s^{\mathrm{i}}_1} \mathcal{B}(\mathfrak{l}_1, s^{\mathrm{i}}_1, s^{\mathrm{f}}_1) \tag{8.47}$$

By inserting the bound from Lemma 8.4 in (8.47), we obtain a bound on $\mathcal{C}_t(\sigma_l, \mathfrak{L})$ depending on the vertex time-coordinates $s^{\mathrm{i}}, s^{\mathrm{f}}$. In the next bound, Lemma 8.5, we simply integrate out these coordinates. To describe the result, it is convenient to introduce some taylor-made notation. Let the times $(t_1, \dots, t_{2n})$ be the time-coordinates of σ_l. We will now specify the effective dynamics between each of those times, depending on the vertex partition $\mathfrak{L}$.

- If the times t_i and t_{i+1} belong to the vertex set of the same vertex, then

$$\tilde{\mathcal{H}}^{\gamma}_{t_{i+1}, t_i} := \tilde{\mathcal{G}}^{\gamma}_{t_{i+1} - t_i}, \qquad \text{with } \tilde{\mathcal{G}}^{\gamma}_t := \mathrm{e}^{-\frac{t}{3\tau}} \tilde{\mathcal{U}}^{\gamma}_t \tag{8.48}$$

- If the times t_i and t_{i+1} belong to different vertices, then

$$\tilde{\mathcal{H}}^{\gamma}_{t_{i+1}, t_i} := \tilde{\mathcal{Z}}^{\tau,\gamma}_{t_{i+1} - t_i} \tag{8.49}$$

The idea of this distinction is clear: Within a dressed vertex, we get additional decay from the short diagrams; this is the origin of the exponential decay $\mathrm{e}^{-\frac{t}{3\tau}}$ in $\tilde{\mathcal{G}}^{\gamma}_t$. Between the vertices, we encounter the cutoff reduced evolution $\mathcal{Z}^{\tau}_t$, as already visible in (8.47). Moreover, we get an additional small factor for each dressed vertex. To make this explicit, we define

$$|\mathfrak{L}|_{\text{dressed}} := \#\{\text{dressed } \mathfrak{l}_k\} \qquad (= \text{number of dressed vertices in the vertex partition } \mathfrak{L}) \tag{8.50}$$

Lemma 8.5. *Let the operators $\tilde{\mathcal{H}}^{\gamma}_{t_i, t_{i+1}}$ be defined as above, depending on the diagram σ_l and the vertex partition $\mathfrak{L}$. Then, for λ, γ small enough,*

$$\mathcal{C}_t(\sigma_l, \mathfrak{L}) \underset{m.e.}{\leq} \left[(|\lambda|\tau)^2 c(\gamma)\right]^{|\mathfrak{L}|_{\mathit{dressed}}} \tilde{\mathcal{G}}^{\gamma}_{t - t_{2n}} \tilde{\mathcal{I}}_{x_{2n}, l_{2n}} \tilde{\mathcal{H}}^{\gamma}_{t_{2n}, t_{2n-1}} \dots \tilde{\mathcal{H}}^{\gamma}_{t_3 - t_2} \tilde{\mathcal{I}}_{x_2, l_2} \tilde{\mathcal{H}}^{\gamma}_{t_2 - t_1} \tilde{\mathcal{I}}_{x_1, l_1} \tilde{\mathcal{G}}^{\gamma}_{t_1} \tag{8.51}$$

Note that between the times 0 and t_1, we always (for each vertex partition) put $\tilde{\mathcal{G}}^{\gamma}_{t_1}$. This is because either $t_1 = 0$, in which case $\tilde{\mathcal{G}}^{\gamma}_{t_1} = 1$, or t_1 belongs to a dressed vertex whose initial time coordinate, s^{i}_1, is fixed to be $s^{\mathrm{i}}_1 = 0$. The same remark applies between the times t_n and t.

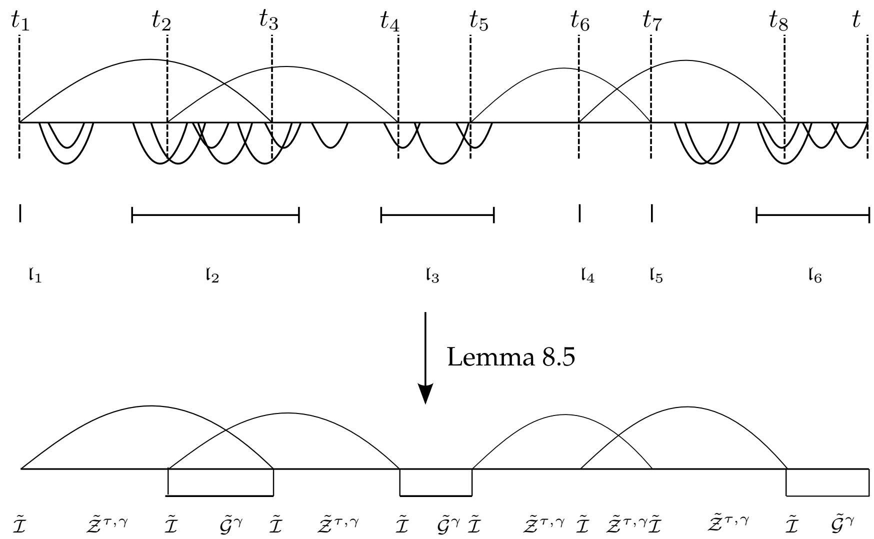


Figure 12: Consider the long diagram $\sigma_l \in \Sigma_{[0,t]}(>\tau)$ with $|\sigma| = 4$ shown above. In the upper figure, we show a short diagram σ such that $\sigma_l \cup \sigma$ is irreducible in $[0,t]$. The corresponding vertex partition $\mathfrak{L} = \{\mathfrak{l}_1, \dots, \mathfrak{l}_6\}$ is indicated by vertical lines for the bare vertices $\mathfrak{l}_1, \mathfrak{l}_4, \mathfrak{l}_5$ and horizontal bars for the dressed vertices $\mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_6$. In the picture below we suggest the representation that emerges after applying Lemma 8.5: There are no vertex time coordinates any more. The time-coordinates of the long diagrams correspond to operators $\tilde{\mathcal{I}}$. The intervals between time-coordinates of the long diagrams correspond to operators $\tilde{\mathcal{Z}}^{\tau,\gamma}$ or $\tilde{\mathcal{G}}^{\gamma}$. The intervals corresponding to $\tilde{\mathcal{G}}^{\gamma}$ are those which in the upper picture belong entirely to the domain of a short diagram.

Proof. The proof starts from the representation of $\mathcal{C}_t(\sigma_l, \mathfrak{L})$ in (8.47) and the bound for the vertex operators $\mathcal{B}(\mathfrak{l}_k, s^{\mathrm{i}}_k, s^{\mathrm{f}}_k)$ given in Lemma 8.4. Then we integrate out the $s^{\mathrm{i}}_k, s^{\mathrm{f}}_k$-coordinates for the dressed vertices $\mathfrak{l}_k$. The main tool in doing so is the fast decay of the function $w(\cdot)$, as follows from (8.45).

We consider a simple example. Take $t_1 = 0$ and $t_{2n} = t$ and let $|\mathfrak{L}| = 1$, i.e. there is one vertex $\mathfrak{l}$. It follows that $\mathfrak{l}$ is dressed and $S(\mathfrak{l}) = \{t_2, \dots, t_{2n-1}\}$. In this case, formula (7.45) reads

$$\mathcal{C}_t(\sigma_l, \mathfrak{L}) = \int_{\substack{0 < s^{\mathrm{i}} < t_2 \\ t_{2n-1} < s^{\mathrm{f}} < t}} \mathrm{d}s^{\mathrm{i}} \mathrm{d}s^{\mathrm{f}} \mathcal{I}_{x_{2n}, l_{2n}} \mathcal{Z}^{\tau}_{t-s^{\mathrm{f}}} \mathcal{B}(\mathfrak{l}, s^{\mathrm{i}}, s^{\mathrm{f}}) \mathcal{Z}^{\tau}_{s^{\mathrm{i}}-t_1} \mathcal{I}_{x_1, l_1} \tag{8.52}$$

and the bound in Lemma 8.4 is

$$\mathcal{B}(\mathfrak{l}, s^{\mathrm{i}}, s^{\mathrm{f}}) \underset{m.e.}{\leq} w(s^{\mathrm{f}} - s^{\mathrm{i}}) \tilde{\mathcal{U}}^{\gamma}_{s^{\mathrm{f}}-t_{2n-1}} \times \underbrace{\tilde{\mathcal{I}}_{x_{2n-1}, l_{2n-1}} \tilde{\mathcal{U}}^{\gamma}_{t_{2n-1}-t_{2n-2}} \dots \tilde{\mathcal{U}}^{\gamma}_{t_3 - t_2} \tilde{\mathcal{I}}_{x_2, l_2}}_{\tilde{\mathcal{A}}^{\gamma}} \times \tilde{\mathcal{U}}^{\gamma}_{t_2 - s^{\mathrm{i}}} \tag{8.53}$$

where the operator $\tilde{\mathcal{A}}^{\gamma}$ is defined as the 'interior part' of the vertex operator. The sole property of $\tilde{\mathcal{A}}^{\gamma}$ that is relevant for the present argument is that

$$\tilde{\mathcal{A}}^{\gamma} \underset{m.e.}{\leq} \mathrm{e}^{c(\gamma,\lambda)t} \tilde{\mathcal{A}}^{\frac{\gamma}{2}} \tag{8.54}$$

as follows from the definition of $\tilde{\mathcal{U}}^{\gamma}_t$ and the bound (8.16). From (8.53), (8.54) and (8.18, 8.19) , we obtain

$$\mathcal{C}_t(\sigma_l, \mathfrak{L}) \underset{m.e.}{\leq} \mathrm{e}^{-\frac{(t_{2n-1}-t_2)}{3\tau}} \int_{\substack{0 < s^{\mathrm{i}} < t_2 \\ t_{2n-1} < s^{\mathrm{f}} < t}} \mathrm{d}s^{\mathrm{i}} \mathrm{d}s^{\mathrm{f}}\, w(s^{\mathrm{f}} - s^{\mathrm{i}})\, \mathrm{e}^{\frac{s^{\mathrm{f}}-s^{\mathrm{i}}}{2\tau}} (c(\gamma))^2 \tag{8.55}$$
$$\tilde{\mathcal{I}}_{x_{2n}, l_{2n}} \tilde{\mathcal{Z}}^{\tau, \frac{\gamma}{2}}_{t_{2n}-t_{2n-1}} \tilde{\mathcal{A}}^{\frac{\gamma}{2}} \tilde{\mathcal{Z}}^{\tau, \frac{\gamma}{2}}_{t_2 - t_1} \tilde{\mathcal{I}}_{x_1, l_1}$$

where we have used the decomposition $s^{\mathrm{f}} - s^{\mathrm{i}} = (s^{\mathrm{f}} - t_{2n-1}) + (t_{2n-1} - t_2) + (t_2 - s^{\mathrm{i}})$ and we have chosen λ, γ small enough such that $c(\gamma, \lambda) < 1/(6\tau)$ in (8.54). By a change of integration variables, we find that

$$\int_{\substack{0 < s^{\mathrm{i}} < t_2 \\ t_{2n-1} < s^{\mathrm{f}} < t}} \mathrm{d}s^{\mathrm{i}} \mathrm{d}s^{\mathrm{f}} w(s^{\mathrm{f}} - s^{\mathrm{i}}) \mathrm{e}^{\frac{s^{\mathrm{f}} - s^{\mathrm{i}}}{2\tau}} \leq \int_{\mathbb{R}^+} \mathrm{d}t \, |t| w(t) \mathrm{e}^{\frac{t}{2\tau}} \tag{8.56}$$

and we note that this bound remains valid if, in the integration domain on the LHS, we replaced 0 by a smaller number, or t_{2n} by a larger number. Hence, by the bound (8.45), we obtain

$$\mathcal{C}_t(\sigma_l, \mathfrak{L}) \underset{m.e.}{\leq} [c(\gamma)]^2 (|\lambda|\tau)^2 \mathrm{e}^{-\frac{(t_{2n-1} - t_2)}{3\tau}} \mathcal{I}_{x_{2n}, l_{2n}} \tilde{\mathcal{Z}}^{\tau,\gamma}_{t_{2n} - t_{2n-1}} \tilde{\mathcal{A}}^{\gamma} \tilde{\mathcal{Z}}^{\tau,\gamma}_{t_2 - t_1} \tilde{\mathcal{I}}_{x_1, l_1} \tag{8.57}$$

where the constant that originates from the RHS of the bound (8.45) has been absorbed in $c(\gamma)$. The bound (8.57) is indeed (8.51) for our special choice of $\mathfrak{L}$ in which $|\mathfrak{L}|_{\text{dressed}} = 1$. To obtain the general bound, one repeats the above calculation for each dressed vertex. These calculations can be performed completely independently of each other, as is visible from the remark below (8.56).

□

In Lemma 8.5, the bound depends on $\mathfrak{L}$ through $\tilde{\mathcal{H}}^{\gamma}$, see (8.48) and (8.49). The next step is to sum over $\mathfrak{L}$. First, we weaken our bound in (8.51) to be valid for all $\mathfrak{L}$, such that the sum over $\mathfrak{L}$ amounts to counting all possible $\mathfrak{L} \sim \sigma_l$. By "weakening the bound", we mean that we bound some of the operators $\tilde{\mathcal{G}}^{\gamma}_t$ by $\tilde{\mathcal{Z}}^{\tau,\gamma}_t$. This can always be done, since, for λ small enough,

$$\tilde{\mathcal{G}}^{\gamma}_t \underset{m.e.}{\leq} \tilde{\mathcal{Z}}^{\tau,\gamma}_t \tag{8.58}$$

with $\mathcal{G}^{\gamma}_t$ as in (8.48) (in fact, $\tilde{\mathcal{G}}^{\gamma}_t$ is smaller than the second term of $\tilde{\mathcal{Z}}^{\tau,\gamma}_t$, see (8.11)). Let $\sigma_1, \ldots, \sigma_m$ be the decomposition of σ_l into irreducible components and let s_{2i-1}, s_{2i} be the boundaries of the domain of σ_i. These times s_i should not be confused with the vertex time-coordinates $\underline{s}^{\mathrm{i}}, \underline{s}^{\mathrm{f}}$ that were employed in an earlier stage of our analysis. In particular, the times $s_{2i-1}, s_{2i}, i = 1, \ldots m$, are a subset of the times $t_i, i = 1, \ldots, 2n$. The central remark is that

For any i, the times s_{2i}, s_{2i+1} belong to the same vertex for all *vertex partitions $\mathfrak{L} \sim \sigma_l$.*

Indeed, since the interval $[s_{2i}, s_{2i+1}]$ is not in the domain of σ_l, it must be in the domain of any short diagram contributing to $\mathcal{C}_t(\sigma_l)$, or, in other words, any vertex partition $\mathfrak{L} \sim \sigma_l$ must contain a vertex whose vertex set contains both s_{2i}, s_{2i+1}. Consequently, the operators $\tilde{\mathcal{H}}^{\gamma}_{s_{2i}, s_{2i+1}}$ in (8.51) are always (i.e., for each compatible vertex partition) equal to $\tilde{\mathcal{G}}^{\gamma}_{s_{2i}, s_{2i+1}}$, and we will not replace them. However, we replace all other $\tilde{\mathcal{H}}^{\gamma}_{t_j, t_{j+1}}$, i.e. those with the property that the times t_j, t_{j+1} are in the domain of the same irreducible component of σ_l, by $\tilde{\mathcal{Z}}^{\tau,\gamma}_{t_{j+1} - t_j}$.

This procedure is illustrated in Figure 13.

After this replacement, the operator part of the resulting expression is independent of $\mathfrak{L}$, and we can perform the sum over $\mathfrak{L} \sim \sigma_l$ by estimating

$$\sum_{\mathfrak{L} \sim \sigma_l} \left[(|\lambda|\tau)^2 c(\gamma) \right]^{|\mathfrak{L}|_{\text{dressed}}} \leq (|\lambda|\tau)^{2v(\sigma_l)} c(\gamma)^{|\sigma_l|}, \qquad \text{for } |\lambda|\tau \leq 1 \tag{8.59}$$

with

$$v(\sigma_l) := \min_{\mathfrak{L} \sim \sigma_l} |\mathfrak{L}|_{\text{dressed}} \tag{8.60}$$

To obtain (8.59), one uses that

$$\#\{\mathfrak{L} \sim \sigma_l\} \leq 4^{2|\sigma_l| - 1} \tag{8.61}$$

Indeed, $2^{2|\sigma_l|-1}$ is the number of ways to partition the time-coordinates into vertex sets. The extra factor 2 for each vertex takes into account the choice bare/dressed.

We have thus arrived at the following lemma

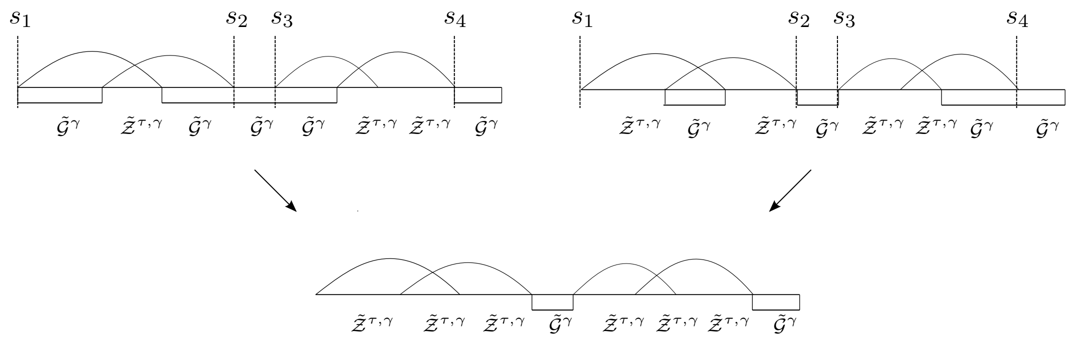


Figure 13: Consider the long diagram $\sigma_l \in \Sigma_{[0,t]}(>\tau)$ with $|\sigma| = 4$ shown above. It has two irreducible components with domains $[s_1, s_2]$ and $[s_3, s_4]$. In the upper figures, two different vertex partitions (compatible with σ_l) are shown together with their respective bounds, obtained in Lemma 8.5. These bounds are represented by the operators $\tilde{\mathcal{G}}^\gamma$ and $\tilde{\mathcal{Z}}^{\tau,\gamma}$, as in Figure 12, except for the fact that we omit the operators $\tilde{\mathcal{I}}$ corresponding to the time coordinates of σ_l. In the lower figure, we show the (weaker) bound that gives rise to Lemma 8.6. To establish this weaker bound, we replace the $\tilde{\mathcal{G}}^\gamma$ that are 'bridged' by the long diagram by $\tilde{\mathcal{Z}}^{\tau,\gamma}$.

Lemma 8.6. *Let s_{2i-2}, s_{2i} be the boundaries of the domain of σ_i, the i'th irreducible component of σ_l. Then, for λ, γ small enough,*

$$\mathcal{C}_t(\sigma_l) \underset{m.e.}{\leq} (|\lambda|\tau)^{2v(\sigma_l)} \tilde{\mathcal{G}}^\gamma_{t-s_{2m}} \tilde{\mathcal{E}}^\gamma(\sigma_m) \tilde{\mathcal{G}}^\gamma_{s_{2m-1}-s_{2m-2}} \tilde{\mathcal{E}}^\gamma(\sigma_{m-1}) \dots \tilde{\mathcal{E}}^\gamma(\sigma_1) \tilde{\mathcal{G}}^\gamma_{s_1} \tag{8.62}$$

where, for an irreducible diagram σ with $|\sigma| = p$,

$$\tilde{\mathcal{E}}^\gamma(\sigma) := [c(\gamma)]^{|\sigma|} \tilde{\mathcal{I}}_{x_{2p}(\sigma), l_{2p}(\sigma)} \tilde{\mathcal{Z}}^{\tau,\gamma}_{t_{2p}(\sigma)-t_{2p-1}(\sigma)} \dots \tilde{\mathcal{Z}}^{\tau,\gamma}_{t_2(\sigma)-t_1(\sigma)} \tilde{\mathcal{I}}_{x_1(\sigma), l_1(\sigma)} \tag{8.63}$$

with $v(\sigma_l)$ as defined above.

Note that $v(\sigma_l)$ is actually the number of factors $\tilde{\mathcal{G}}^\gamma_u$ in the expression (8.62) for which $u \neq 0$ (u can be zero only for the rightmost and leftmost $\tilde{\mathcal{G}}^\gamma_u$). Or, alternatively,

$$v(\sigma_l) = \#\{\text{irreducible components in } \sigma_l\} - 1 + 1_{t_{2n} \neq t} + 1_{t_1 \neq 0} \tag{8.64}$$

8.7 Bounds on $\mathcal{R}_{ex}(z)$ in terms of $\tilde{\mathcal{E}}^\gamma(\sigma)$

To realize why the bound (8.62) in Lemma 8.6 is useful, we recall that our aim is to calculate $\mathcal{R}_{ex}(z)$, given by (see (7.33))

$$\mathcal{R}_{ex}(z) = \int_{\mathbb{R}^+} \mathrm{d}t \, \mathrm{e}^{-tz} \int_{\Sigma_{[0,t]}(>\tau)} \mathrm{d}\sigma \zeta(\sigma) \mathcal{C}_t(\sigma) \; [1 + \delta(t_1(\sigma_l))] \; \left[1 + \delta(t_{2|\sigma_l|}(\sigma_l) - t)\right] \tag{8.65}$$

We calculate $\mathcal{R}_{ex}(z)$ by replacing the integral over diagrams by an integral over sequences of irreducible diagrams, as we did in (7.23), i.e.,

$$\int_{\Sigma_{[0,t]}(>\tau)} \mathrm{d}\sigma \dots = \sum_{n \geq 1} \int_{0 \leq s_1 < \dots < s_{2n} \leq t} \prod_{j=1}^n \left(\int_{\Sigma_{[s_{2j-1}, s_{2j}]}(>\tau, \mathrm{ir})} \mathrm{d}\sigma_j \right) \dots \tag{8.66}$$

Using the bound (8.62), we obtain, with the shorthand $z_r := \operatorname{Re} z$,

$$\begin{aligned} \mathcal{R}_{ex}(z) \quad & \underset{m.e.}{\leq} \quad \sum_{n \geq 0} (1 + \mathcal{R}_{\tilde{\mathcal{G}}}(z_r)) \mathcal{R}_{\tilde{\mathcal{E}}}(z_r) \left(\mathcal{R}_{\tilde{\mathcal{G}}}(z_r) \mathcal{R}_{\tilde{\mathcal{E}}}(z_r) \right)^n (1 + \mathcal{R}_{\tilde{\mathcal{G}}}(z_r)) \\ & = \quad (1 + \mathcal{R}_{\tilde{\mathcal{G}}}(z_r)) \mathcal{R}_{\tilde{\mathcal{E}}}(z_r) \left(1 - \mathcal{R}_{\tilde{\mathcal{G}}}(z_r) \mathcal{R}_{\tilde{\mathcal{E}}}(z_r) \right)^{-1} (1 + \mathcal{R}_{\tilde{\mathcal{G}}}(z_r)) \end{aligned} \tag{8.67}$$

where, for $\mathrm{Re}\, z$ large enough,

$$\mathcal{R}_{\tilde{\mathcal{G}}}(z) \quad := \quad (\tau|\lambda|)^2 \int_{\mathbb{R}^+} \mathrm{d}t\, \mathrm{e}^{-tz} \tilde{\mathcal{G}}_t^{\gamma} \tag{8.68}$$

$$\mathcal{R}_{\tilde{\mathcal{E}}}(z) \quad := \quad \int_{\mathbb{R}^+} \mathrm{d}t\, \mathrm{e}^{-tz} \int\limits_{\Sigma_{[0,t]}(>\tau,\mathrm{ir})} \mathrm{d}\sigma \zeta(\sigma) \tilde{\mathcal{E}}^{\gamma}(\sigma) \tag{8.69}$$

Since $\tilde{\mathcal{G}}_t^{\gamma}$ is known explicitly, the only task that remains is to study $\mathcal{R}_{\tilde{\mathcal{E}}}(z)$. This study is undertaken in Section 9.

9 The renormalized model

In this section, we prove Lemma 6.5, thereby concluding the proof of our main result, Theorem 3.3. We briefly recall the logic of our proof. As announced in Section 5.4.1, we analyze $\mathcal{Z}_t^{\mathrm{ir}}$ and $\mathcal{Z}_t$ through a renormalized perturbation series, where the short diagrams have already been resummed. However, we do not study the Laplace transform of irreducible diagrams (defined in (7.20))

$$\mathcal{R}^{\mathrm{ir}}(z) = \int\limits_{\mathbb{R}^+} \mathrm{d}t\, \mathrm{e}^{-tz} \mathcal{Z}_t^{\mathrm{ir}} = \int\limits_{\mathbb{R}^+} \mathrm{d}t\, \mathrm{e}^{-tz} \int\limits_{\Sigma_{[0,t]}(\mathrm{ir})} \mathrm{d}\sigma\, \zeta(\sigma) \mathcal{V}_t(\sigma), \tag{9.1}$$

directly, but rather the Laplace transform of irreducible renormalized diagrams (defined in Lemma 8.6 and Section 8.7)

$$\mathcal{R}_{\tilde{\mathcal{E}}}(z) = \int\limits_{\mathbb{R}^+} \mathrm{d}t\, \mathrm{e}^{-tz} \int\limits_{\Sigma_{[0,t]}(>\tau,\mathrm{ir})} \mathrm{d}\sigma \zeta(\sigma) \tilde{\mathcal{E}}^{\gamma}(\sigma). \tag{9.2}$$

Although the quantities (9.2) and (9.1) are not equal, we will argue below (in the proof of Lemma 6.5 starting from Lemma 9.1) that good bounds on $\mathcal{R}_{\tilde{\mathcal{E}}}(z)$ yield good bounds on $\mathcal{R}^{\mathrm{ir}}(z)$ and hence also on $\mathcal{R}(z)$. The reason that the expression (9.1) itself cannot be bounded by an integral over long irreducible diagrams, is the fact that an irreducible diagram in the interval $[0,t]$ does not necessarily contain an irreducible *long* subdiagram in the interval $[0,t]$. Indeed, Lemma 8.6 decomposes the domain of an irreducible diagram into domains of long, irreducible subdiagrams and intermediate intervals. These remaining intervals give rise to operators $\tilde{\mathcal{G}}_t^{\gamma}$, which are easily dealt with, as we will see below, in the proof of Lemma 6.5, since they originate from short diagrams and therefore have good decay properties.

Nevertheless, we clearly see the similarity between (9.1) and (9.2). To highlight this similarity, we write the inverse Laplace transform of $\mathcal{R}_{\tilde{\mathcal{E}}}(z)$: For $r>0$ large enough, we have

$$\frac{1}{2\pi i} \int\limits_{r+\mathrm{i}\mathbb{R}} \mathrm{d}z\, \mathrm{e}^{tz} \mathcal{R}_{\tilde{\mathcal{E}}}(z) = \int\limits_{\Sigma_{[0,t]}(>\tau,\mathrm{ir})} \mathrm{d}\sigma\, c(\gamma)^{|\sigma|} \zeta(\sigma)\, \tilde{\mathcal{I}}_{x_{2n},l_{2n}} \tilde{\mathcal{Z}}^{\tau,\gamma}_{t_{2n}-t_{2n-1}} \ldots \tilde{\mathcal{I}}_{x_2,l_2} \tilde{\mathcal{Z}}^{\tau,\gamma}_{t_2-t_1} \tilde{\mathcal{I}}_{x_1,l_1} \tag{9.3}$$

where $\underline{t}, \underline{x}, \underline{l}$ are the coordinates of σ and, since σ is irreducible in $[0,t]$, $t_1=0$ and $t_{2n}=t$. The inverse Laplace transform of $\mathcal{R}^{\mathrm{ir}}(z)$, i.e. $\mathcal{Z}_t^{\mathrm{ir}}$, is

$$\mathcal{Z}_t^{\mathrm{ir}} = \int\limits_{\Sigma_{[0,t]}(\mathrm{ir})} \mathrm{d}\sigma\, \zeta(\sigma) \mathcal{I}_{x_{2n},l_{2n}} \mathcal{U}_{t_{2n}-t_{2n-1}} \ldots \mathcal{I}_{x_2,l_2} \mathcal{U}_{t_2-t_1} \mathcal{I}_{x_1,l_1} \tag{9.4}$$

where $\underline{t}, \underline{x}, \underline{l}$ have the same meaning as above. Thus, the perturbation series in (9.3) is indeed a renormalized version of (9.4). The diagrams are constrained to be long, and the short diagrams have been absorbed into the 'dressed free propagator' $\tilde{\mathcal{Z}}_t^{\tau,\gamma}$. This point of view has also been stressed in Section 5.5. Observe, however, that $\mathcal{Z}_t^{\tau,\gamma}$ depends on the positive parameter γ, whereas there is no such dependence in (9.1).

The following lemma is our main result on $\mathcal{R}_{\tilde{\mathcal{E}}}(z)$.

Lemma 9.1. *Recall that $\mathcal{R}_{\tilde{\mathcal{E}}}(z)$ depends on γ, because $\tilde{\mathcal{E}}^{\gamma}(\cdot)$ does. One can choose γ such that there are positive constants $g_{ex} > 0$ and $\delta_{ex} > 0$, such that*

$$\sup_{|\mathrm{Im}\,\kappa_{R,L}|\leq\delta_{ex},\mathrm{Re}\,z\geq-\lambda^2 g_{ex}} \|\mathcal{J}_\kappa \mathcal{R}_{\tilde{\mathcal{E}}}(z)\mathcal{J}_{-\kappa}\| = o(\lambda^2), \qquad \text{as } \lambda \searrow 0 \tag{9.5}$$

The main tools in the proof of Lemma 9.1 will be the exponential decay of the 'renormalized correlation function', which follows from the bounds on $\mathcal{Z}_t^{\tau}$ stated in Lemma 6.2, and the strategy for integrating over diagrams presented in Lemma 8.2. With Lemma 9.1 at hand, the proof of Lemma 6.5 is immediate.

Proof of Lemma 6.5 We only need to prove Statement 2) since Statement 1) will follow by a remark analogous to that in the proof of Statement 1) of Lemma 6.1. Clearly, for λ small enough,

$$\sup_{|\mathrm{Im}\,\kappa_{R,L}|\leq\gamma,\mathrm{Re}\,z\geq-\frac{1}{4\tau}} \left\|\mathcal{J}_\kappa \mathcal{R}_{\tilde{\mathcal{G}}}(z)\mathcal{J}_{-\kappa}\right\| \leq O(\lambda), \qquad \text{as } \lambda \searrow 0 \tag{9.6}$$

This follows from the properties of $\tilde{\mathcal{U}}_t^{\gamma}$, see e.g. the proof of Lemma 6.1, and the definition of $\tilde{\mathcal{G}}_t^{\gamma}$, see (8.48). Next, we remark that

$$\|\mathcal{J}_\kappa \mathcal{R}_{ex}(z)\mathcal{J}_{-\kappa}\| \leq \|1+\mathcal{J}_\kappa \mathcal{R}_{\tilde{\mathcal{G}}}(z_r)\mathcal{J}_{-\kappa}\|^2 \|\mathcal{J}_\kappa \mathcal{R}_{\tilde{\mathcal{E}}}(z_r)\mathcal{J}_{-\kappa}\| \| \left(1-\mathcal{J}_\kappa \mathcal{R}_{\tilde{\mathcal{G}}}(z_r)\mathcal{R}_{\tilde{\mathcal{E}}}(z_r)\mathcal{J}_{-\kappa}\right)^{-1} \| \tag{9.7}$$

with $z_r = \mathrm{Re}\, z$. This follows from the bound (8.67), the fact that $\mathcal{J}_\kappa \mathcal{J}_{-\kappa} = 1$, and the implication (8.6) (which allows to pass from ' $\underset{m.e.}{\leq}$ ' to an inequality between norms).

Hence, Statement 2) follows by plugging the bounds of Lemma 9.1 and (9.6) into the the RHS of (9.7). □

9.1 Bound on the renormalized correlation function

In this section, we prove Lemma 9.2, which establishes (as its first claim) the property (5.23) with Λ_t replaced by $\mathcal{Z}_t^{\tau}$. Indeed, in Section 6.1, we argued that $\mathcal{Z}_t^{\tau}$ is very close to Λ_t, and this was made explicit in Lemma 6.2. Let

$$h(t) := \lambda^2 c_h \sup_{x\in\mathbb{Z}^d} \begin{cases} \mathrm{e}^{-(1/2)g_{\mathrm{R}} t} & |x|/t \leq v^* \\ \\ |\psi(x,t)| & |x|/t \geq v^* \end{cases} \tag{9.8}$$

with the velocity v^* and decay rate g_{R} as in Lemma 5.1 and the constant c_h chosen such that

$$\lambda^2 \sup_x |\psi(x,t)| \leq h(t), \qquad \text{for } t > \tau \tag{9.9}$$

Lemma 5.1 ensures that such a choice is possible.

Lemma 9.2. *There are positive constants $\delta_r > 0$ and $g_r > 0$ such that, for all $\gamma < \delta_r$, λ small enough, and $\kappa \equiv (\kappa_L, \kappa_R)$ satisfying $|\mathrm{Im}\,\kappa_{R,L}| \leq \frac{\delta_r}{2}$,*

$$\lambda^2 |\psi(x'_{l_2} - x_{l_1}, t)| \times \left\| (\mathcal{J}_\kappa \tilde{\mathcal{Z}}_t^{\tau,\gamma} \mathcal{J}_{-\kappa})_{x_L,x_R;x'_L,x'_R} \right\| \leq h(t)\mathrm{e}^{-\lambda^2 g_r t}, \qquad \textit{for} \qquad l_1, l_2 \in \{L, R\} \tag{9.10}$$

and

$$\left\| (\mathcal{J}_\kappa \tilde{\mathcal{Z}}_t^{\tau,\gamma} \mathcal{J}_{-\kappa})_{x_L,x_R;x'_L,x'_R} \right\| \leq C\mathrm{e}^{c(\gamma,\lambda)t} \tag{9.11}$$

This lemma is derived from the bound (6.7) in Lemma 6.2 in a way that is completely analogous to the proof of (5.23) starting from (4.55), as outlined in Section 5.4.1. The only difference is that in Lemma 9.2, we allow for a small blowup in space given by the multiplication operator $\mathcal{J}_\kappa$.

For future use, we also define

$$h_\tau(t) := 1_{|t|\geq\tau} h(t) \tag{9.12}$$

and we note that

$$\|h_\tau\|_1 := \int_{\mathbb{R}^+} h_\tau(t) = o(\lambda^2), \qquad \text{as } \lambda \searrow 0 \tag{9.13}$$

since $\|h\|_1 < \infty$ and $\tau(\lambda) \to \infty$ as $\lambda \searrow 0$.

9.2 Sum over non-minimally irreducible diagrams

In a first step towards performing the integral in (9.2), we reduce the integral over irreducible diagrams to an integral over minimally irreducible diagrams. Indeed, since any diagram that is irreducible in I has a minimally irreducible (in I) subdiagram, we have, for any positive function F,

$$\int_{\Sigma_I(\mathrm{ir},>\tau)} \mathrm{d}\sigma F(\sigma) \leq \int_{\Sigma_I(\mathrm{mir},>\tau)} \mathrm{d}\sigma \left(F(\sigma) + \int_{\Sigma_I(>\tau)} \mathrm{d}\sigma' F(\sigma \cup \sigma') \right) \tag{9.14}$$

The first term between brackets on the RHS corresponds to the minimally irreducible diagrams on the LHS. The second term contains the integration over 'additional' diagrams σ'. The integration over these diagrams is unconstrained since $\sigma \cup \sigma'$ is irreducible in I for any σ', provided that σ is irreducible in I. This is also explained and used in Appendix D: see (D.5) and (D.6).

Lemma 9.3 shows that such an integration over unconstrained long diagrams yields a factor $\exp\{c'(\gamma,\lambda)|I|\}$, with the generic constant $c'(\gamma,\lambda)$ as introduced in Section 8.1.

Lemma 9.3. *For λ, γ small enough,*

$$\int_{\Sigma_{[0,t]}(>\tau,\mathrm{ir})} \mathrm{d}\sigma\, \tilde{\mathcal{E}}^{\gamma}(\sigma)|\zeta(\sigma)| \underset{m.e.}{\leq} \mathrm{e}^{c'(\gamma,\lambda)t} \int_{\Sigma_{[0,t]}(>\tau,\mathrm{mir})} \mathrm{d}\sigma\, |\zeta(\sigma)| \tilde{\mathcal{E}}^{\frac{\gamma}{2}}(\sigma) \tag{9.15}$$

Proof. By formula (9.14) (applied in the case where $F(\sigma)$ is a matrix element of the operator $|\zeta(\sigma)|\tilde{\mathcal{E}}^{\gamma}(\sigma)$), we have that

$$\int_{\Sigma_{[0,t]}(>\tau,\mathrm{ir})} \mathrm{d}\sigma\, \tilde{\mathcal{E}}^{\gamma}(\sigma)|\zeta(\sigma)| \underset{m.e.}{\leq} \int_{\Sigma_{[0,t]}(>\tau,\mathrm{mir})} \mathrm{d}\sigma\, \tilde{\mathcal{E}}^{\gamma}(\sigma)|\zeta(\sigma)| + \int_{\Sigma_{[0,t]}(>\tau,\mathrm{mir})} \mathrm{d}\sigma \int_{\Sigma_{[0,t]}(>\tau)} \mathrm{d}\sigma'\, \tilde{\mathcal{E}}^{\gamma}(\sigma \cup \sigma')|\zeta(\sigma \cup \sigma')| \tag{9.16}$$

First, we bound

$$\int_{\Sigma_{[0,t]}(>\tau)} \mathrm{d}\sigma'\, \tilde{\mathcal{E}}^{\gamma}(\sigma \cup \sigma')|\zeta(\sigma \cup \sigma')| \tag{9.17}$$

with σ fixed. To perform the integral over σ' in (9.17), we recall that $\tilde{\mathcal{E}}^{\gamma}(\cdot)$ consists of products of the operators $\tilde{\mathcal{I}}_{x_i,l_i}$ and $\mathcal{Z}^{\tau,\gamma}_{t_{i+1}-t_i}$. Hence, by Lemma 8.2 with $\tilde{\mathcal{U}}^{\gamma}_t$ replaced by $\tilde{\mathcal{Z}}^{\tau,\gamma}_t$, we can sum over the $\underline{x}, \underline{l}$-coordinates of σ' and multiply the $\tilde{\mathcal{Z}}^{\tau,\gamma}_{t_{i+1}-t_i}$ operators using the bound (8.17). This yields

$$(9.17) \underset{m.e.}{\leq} \tilde{\mathcal{E}}^{\frac{\gamma}{2}}_t(\sigma)|\zeta(\sigma)|\, \mathrm{e}^{c(\gamma,\lambda)t} \int_{\Pi_T \Sigma_t(>\tau)} \mathrm{d}[\sigma'] c(\gamma)^{|\sigma'|} \sup_{\underline{x}(\sigma'),\underline{l}(\sigma')} |\zeta(\sigma')|. \tag{9.18}$$

The integral on the RHS of (9.18) is estimated as

$$\int_{\Pi_T \Sigma_t(>\tau)} \mathrm{d}[\sigma'] c(\gamma)^{|\sigma'|} \sup_{\underline{x}(\sigma'),\underline{l}(\sigma')} |\zeta(\sigma')| \quad \leq \quad \mathrm{e}^{c(\gamma)\|h_\tau\|_1 t} - 1 \tag{9.19}$$

with h_τ as defined in (9.12). This follows from the bound (D.8) (integral over unconstrained diagrams) in Appendix D. To bound the first term in (9.16), we dominate

$$\tilde{\mathcal{E}}^{\gamma}(\sigma) \underset{m.e.}{\leq} \mathrm{e}^{c'(\gamma,\lambda)t}\, \tilde{\mathcal{E}}^{\frac{\gamma}{2}}(\sigma) \tag{9.20}$$

The lemma follows by inserting the bounds (9.20) and (9.18, 9.19) in (9.16) and using that $c(\gamma)\|h_\tau\|_1 = c'(\gamma,\lambda)$ since $\|h_\tau\|_1 = o(\lambda^2)$. □

9.3 Sum over minimally irreducible diagrams

In this section, we perform the integral

$$\int_{\Sigma_t(>\tau,\mathrm{mir})} \mathrm{d}\sigma \, |\zeta(\sigma)| \tilde{\mathcal{E}}^{\gamma}(\sigma) \tag{9.21}$$

that appears on the RHS of the bound in Lemma 9.3 (upon replacing $\gamma \to \gamma/2$), and we prove that it is exponentially decaying in time with decay rate $O(\lambda^2)$, for well-chosen γ and λ small enough, depending on γ. It is in this place that we use the decay property of the renormalized correlation function that was stated in Lemma 9.2.

The key idea is the following. If σ is a long diagram with $|\sigma| = 1$ consisting of the two triples $(t_i, x_i, l_i)_{i=1}^{2}$, then

$$|\zeta(\sigma)| \tilde{\mathcal{E}}^{\gamma}(\sigma) = c(\gamma) |\psi^{\#}(x_2 - x_1, t_2 - t_1)| \quad \tilde{\mathcal{I}}_{x_2,l_2} \tilde{\mathcal{Z}}^{\tau,\gamma}_{t_2-t_1} \tilde{\mathcal{I}}_{x_1,l_1} \tag{9.22}$$

In this case, we can obviously use Lemma 9.2 to deduce exponential decay in $t_2 - t_1$ of (9.22), uniformly in x_1, x_2, l_1, l_2. In general, there is of course more than one pairing in an irreducible diagram and so one has to 'split' the decay coming from $\tilde{\mathcal{I}}_{x_2,l_2} \tilde{\mathcal{Z}}^{\tau,\gamma}_{t_2-t_1} \tilde{\mathcal{I}}_{x_1,l_1}$ between the different pairings, thus weakening the decay by a factor which can be as high as $|\sigma|$.

However, since we are considering minimally irreducible pairings, there are at most two pairings bridging any given time t', see Figure 14. Hence, one can attempt to split the decay from $\tilde{\mathcal{I}}_{x_2,l_2} \tilde{\mathcal{Z}}^{\tau,\gamma}_{t_2-t_1} \tilde{\mathcal{I}}_{x_1,l_1}$ in half. This can be done and it is described in Lemma 9.4.

For $\sigma \in \Sigma_{[0,t]}$, we define the function

$$H_\tau(\sigma) = \prod_{j=1}^{|\sigma|} h_\tau(v_j - u_j) \tag{9.23}$$

with h_τ as in (9.12). Note that $H_\tau(\sigma)$ depends only on the equivalence class $[\sigma]$, and hence we can write $H_\tau([\sigma]) := H_\tau(\sigma)$.

Lemma 9.4. *Let the positive constants g_r and δ_r be defined as in Lemma 9.2 and choose $\gamma < \delta_r$. Let $\kappa = (\kappa_L, \kappa_R)$ such that $|\mathrm{Im}\, \kappa_{L,R}| \leq \gamma/8$ and fix a long irreducible diagram class $[\sigma] \in \Pi_T \Sigma_{[0,t]}(> \tau, \mathrm{mir})$. Then*

$$\left\| \sum_{\underline{x}(\sigma), \underline{l}(\sigma)} |\zeta(\sigma)| \mathcal{J}_\kappa \tilde{\mathcal{E}}^{\gamma}(\sigma) \mathcal{J}_{-\kappa} \right\| \leq c(\gamma)^{|\sigma|} \mathrm{e}^{c(\gamma,\lambda)t} \mathrm{e}^{-\frac{1}{2}\lambda^2 g_r t} H_\tau([\sigma]) \tag{9.24}$$

Note that the operator between $\| \cdot \|$ depends on the equivalence class $[\sigma]$ only, due to the sum over $\underline{x}(\sigma), \underline{l}(\sigma)$.

Proof. For concreteness, we assume that $|\sigma| = n$ is even, the argument for $|\sigma|$ odd is analogous. We can find σ_1 and σ_2 such that $\sigma_1 \cup \sigma_2 = \sigma$, $|\sigma_1| = |\sigma_2| = n/2$ and σ_1 and σ_2 are ladder diagrams, i.e. their decompositions into irreducible components consists of singletons. To be more concrete, the time-pairs of σ_1 are

$$(t_1, t_3), (t_4, t_7), (t_8, t_{11}) \ldots (t_{2n-4}, t_{2n-1}) \tag{9.25}$$

and those of σ_2 are

$$(t_2, t_5), (t_6, t_9), (t_{10}, t_{13}) \ldots (t_{2n-2}, t_{2n}) \tag{9.26}$$

The possibility of making such a decomposition is a consequence of the structure of minimally irreducible diagrams, as illustrated in Figure 14.

We now estimate the LHS of (9.24) in two ways. In our first estimate, we take the supremum over the $\underline{x}, \underline{l}$-coordinates of σ_2 and we keep those of σ_1. In the second estimate, the roles of σ_1 and σ_2 are reversed.

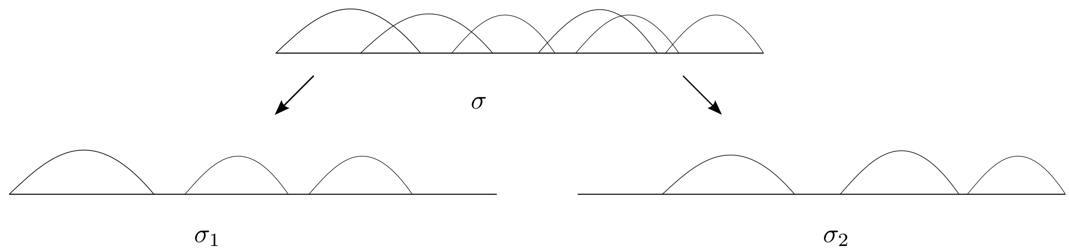


Figure 14: The decomposition of a minimally irreducible diagram σ into two 'ladder diagrams' σ_1 and σ_2. In the upper figure, one can easily check that any point on the (horizontal) time-axis is bridged by at most two pairings.

We estimate

$$\sum_{x(\sigma),l(\sigma)} |\zeta(\sigma)| \tilde{\mathcal{E}}^{\gamma}(\sigma) \tag{9.27}$$

$$= \quad [c(\gamma)]^{|\sigma|} \sum_{x(\sigma_2),l(\sigma_2)} |\zeta(\sigma_2)| \sum_{x(\sigma_1),l(\sigma_1)} |\zeta(\sigma_1)| \left(\tilde{\mathcal{I}}_{x_{2n},l_{2n}} \mathcal{Z}^{\tau,\gamma}_{t_{2n}-t_{2n-1}} \dots \tilde{\mathcal{I}}_{x_2,l_2} \mathcal{Z}^{\tau,\gamma}_{t_2-t_1} \tilde{\mathcal{I}}_{x_1,l_1} \right)$$

$$\underset{m.e.}{\leq} \quad [c(\gamma)]^{|\sigma|} \mathrm{e}^{c(\gamma,\lambda)t} [c(\gamma)]^{|\sigma_2|} \left(\sup_{\underline{x}(\sigma_2),\underline{l}(\sigma_2)} |\zeta(\sigma_2)| \right)$$

$$\sum_{x(\sigma_1),l(\sigma_1)} |\zeta(\sigma_1)| \tilde{\mathcal{Z}}^{\tau,\frac{\gamma}{2}}_{t_{2n}-t_{2n-1}} \tilde{\mathcal{I}}_{x_{2n-1},l_{2n-1}} \tilde{\mathcal{Z}}^{\tau,\frac{\gamma}{2}}_{t_{2n-1}-t_{2n-4}} \dots \tilde{\mathcal{I}}_{x_3,l_3} \tilde{\mathcal{Z}}^{\tau,\frac{\gamma}{2}}_{t_3-t_1} \tilde{\mathcal{I}}_{x_1,l_1})$$

The equality follows from the definition of $\tilde{\mathcal{E}}^{\gamma}(\sigma)$. To obtain the inequality on the third line, we perform the sum over $\underline{x}(\sigma_2), \underline{l}(\sigma_2)$ by the same procedure that was used to obtain (9.18), i.e., by using Lemma 8.2. Since $|\zeta(\sigma_1)|$ factorizes into a function of the pairs in σ_1, the operator part in the last line of (9.27) is a product of two types of terms, namely;

$$\lambda^2 \sum_{\substack{x_{4i}, x_{4i+3} \\ l_{4i}, l_{4i+3}}} |\psi(x_{4i+3}-x_{4i}, t_{4i+3}-t_{4i})| \, \tilde{\mathcal{I}}_{x_{4i+3},l_{4i+3}} \tilde{\mathcal{Z}}^{\tau,\frac{\gamma}{2}}_{t_{4i+3}-t_{4i}} \tilde{\mathcal{I}}_{x_{4i},l_{4i}} \tag{9.28}$$

for $i = 1, \dots, n/2-1$ (and an analogous term where we replace $4i \to 1$ and $4i+3 \to 3$, corresponding to the first pair in σ_1) and

$$\tilde{\mathcal{Z}}^{\tau,\frac{\gamma}{2}}_{t_{4i}-t_{4i-1}} \tag{9.29}$$

for $i = 1, \dots, n/2$.

We note that Lemma 9.2 provides a bound on the matrix elements of these expressions. In particular, we use (9.10) to bound (9.28) and (9.11) to bound (9.29). We obtain, for $|\mathrm{Im}\,\kappa_{L,R}| \leq \frac{\gamma}{4}$,

$$\left(\mathcal{J}_{\kappa}\,(9.28)\,\mathcal{J}_{-\kappa}\right)_{x_L,x_R;x'_L,x'_R} \quad \leq \quad h_{\tau}\left(t_{4i+3}-t_{4i}\right) \mathrm{e}^{-\lambda^2 g_r(t_{4i+3}-t_{4i})} \tag{9.30}$$

$$\left(\mathcal{J}_{\kappa}\,(9.29)\,\mathcal{J}_{-\kappa}\right)_{x_L,x_R;x'_L,x'_R} \quad \leq \quad e^{c(\gamma,\lambda)(t_{4i}-t_{4i-1})} \tag{9.31}$$

By the relation stated in (8.34) and the line following it, we can convert these bounds on the kernels into bounds on the operator norms, yielding, for $|\mathrm{Im}\,\kappa_{L,R}| \leq \frac{1}{8}\gamma$

$$\|\mathcal{J}_{\kappa}\,(9.28)\,\mathcal{J}_{-\kappa}\| \quad \leq \quad c(\gamma) h_{\tau}\left(t_{2i+1}-t_{2i-1}\right) \mathrm{e}^{-\lambda^2 g_r(t_{4i+3}-t_{4i})} \tag{9.32}$$

$$\|\mathcal{J}_{\kappa}\,(9.29)\,\mathcal{J}_{-\kappa}\| \quad \leq \quad c(\gamma) \mathrm{e}^{c(\gamma,\lambda)(t_{4i}-t_{4i-1})} \tag{9.33}$$

and hence, by multiplying these bounds for the operators appearing in (9.27) and using that

$$h_\tau(t_3-t_1)\prod_{i=1}^{n/2-1} h_\tau(t_{4i+3}-t_{4i}) \leq H_\tau(\sigma_1), \qquad \sup_{\underline{x}(\sigma_2),\underline{l}(\sigma_2)} |\zeta(\sigma_2)| \leq H_\tau(\sigma_2), \tag{9.34}$$

(see (9.9)), we arrive at

$$\|\mathcal{J}_\kappa \text{ (9.27) } \mathcal{J}_{-\kappa}\| \leq c(\gamma)^{|\sigma|} H_\tau(\sigma) \mathrm{e}^{-\lambda^2 g_r |\mathrm{Dom}\sigma_1|} \mathrm{e}^{c(\gamma,\lambda)t} \tag{9.35}$$

The claim of the lemma now follows by applying the same bound with the roles of σ_1 and σ_2 swapped, taking the geometric mean of the two bounds and noting that

$$[0,t] \leq |\mathrm{Dom}\sigma_1| + |\mathrm{Dom}\sigma_2| \tag{9.36}$$

□

Next, we use Lemmata 9.3 and 9.4 to prove Lemma 9.1. By these two lemmas, the integral over renormalized irreducible diagrams is reduced to an integral over minimally irreducible equivalence classes $[\sigma]$. Each equivalence class $[\sigma]$ essentially contributes $c(\gamma)^{|\sigma|} H_\tau(\sigma)$ to the integral. Since $H_\tau(\sigma)$ is not exponentially decaying in $\mathrm{Dom}\sigma$, the Laplace transform of $H_\tau(\sigma)$ cannot be continued to negative $\mathrm{Re}\, z$. However, the factor $\mathrm{e}^{-\lambda^2 \frac{1}{2} g_r t}$ in Lemma 9.4 enables us to do such a continuation since the factors $c'(\gamma,\lambda), c(\gamma,\lambda)$ from Lemmata 9.3 and 9.4 can be made smaller than $\lambda^2 \frac{1}{2} g_r$ by first choosing γ small enough, and then adjusting λ.

Proof of Lemma 9.1 We choose γ small enough, as required in the conditions of Lemmata 9.3 and 9.4, and we estimate, for $|\mathrm{Im}\,\kappa_{L,R}| \leq \gamma/8$,

$$\begin{aligned}
\left\| \int_{\Sigma_{[0,t]}(>\tau,\mathrm{ir})} \mathrm{d}\sigma |\zeta(\sigma)| \mathcal{J}_\kappa \tilde{\mathcal{E}}^{\gamma}(\sigma) \mathcal{J}_{-\kappa} \right\| &\leq \mathrm{e}^{c'(\gamma,\lambda)t} \left\| \int_{\Sigma_{[0,t]}(>\tau,\mathrm{mir})} \mathrm{d}\sigma |\zeta(\sigma)| \mathcal{J}_\kappa \tilde{\mathcal{E}}^{\frac{\gamma}{2}}(\sigma) \mathcal{J}_{-\kappa} \right\| \\
&\leq \mathrm{e}^{c'(\gamma,\lambda)t} \int_{\Pi_T \Sigma_{[0,t]}(>\tau,\mathrm{mir})} \mathrm{d}[\sigma] \left\| \sum_{\underline{x}(\sigma),\underline{l}(\sigma)} |\zeta(\sigma)|\, \mathcal{J}_\kappa \tilde{\mathcal{E}}^{\frac{\gamma}{2}}(\sigma) \mathcal{J}_{-\kappa} \right\| \\
&\leq \mathrm{e}^{(c'(\gamma,\lambda)-\frac{1}{4}\lambda^2 g_r)t} \int_{\Pi_T \Sigma_{[0,t]}(>\tau,\mathrm{mir})} \mathrm{d}[\sigma]\, c(\gamma)^{|\sigma|} \mathrm{e}^{-\frac{1}{4}\lambda^2 g_r t} H_\tau(\sigma)
\end{aligned} \tag{9.37}$$

The first inequality is Lemma 9.3, the second inequality uses the definition of the measure $\mathrm{d}\sigma$, and the third inequality follows from Lemma 9.4.

We will now estimate the Laplace transform of the integral in the last line of (9.37). To prove Lemma 9.1, we fix $g_{ex} := \frac{1}{8} g_r$ and we show that one can choose γ such that, for λ small enough and $\mathrm{Re}\, z \geq -\lambda^2 g_{ex}$,

$$\int_{\mathbb{R}^+} \mathrm{d}t\, \mathrm{e}^{-tz} \mathrm{e}^{(c'(\gamma,\lambda)-\frac{1}{4}\lambda^2 g_r)t} \int_{\Pi_T \Sigma_{[0,t]}(>\tau,\mathrm{mir})} \mathrm{d}[\sigma] c(\gamma)^{|\sigma|} \mathrm{e}^{-\frac{1}{4}\lambda^2 g_r t} H_\tau(\sigma) = o(\lambda^2), \qquad \text{as } \lambda \searrow 0 \tag{9.38}$$

Of course, the choice of γ will have to depend on the specific value of $c'(\gamma,\lambda)$. To show (9.38), we first choose γ such that, for λ small enough, we have

$$c'(\gamma,\lambda) - \frac{1}{4}\lambda^2 g_r \leq -\frac{1}{8}\lambda^2 g_r \tag{9.39}$$

with $c'(\gamma,\lambda)$ as in (9.38).

Consequently, we can dominate the factor $\mathrm{e}^{-zt}\mathrm{e}^{(c'(\gamma,\lambda)-\frac{1}{4}\lambda^2 g_r)t}$ by 1 in (9.38). Next, we note that, if σ is minimally irreducible in the interval $[0,t]$, then

$$\sum_{i=1}^{|\sigma|} |v_i - u_i| \leq 2t \tag{9.40}$$

where (u_i, v_i) are the pairs of time-coordinates associated to σ. This follows from the observation that each point in the interval $[0,t]$ is bridged by at most two pairings of σ, see also Figure 14. Consequently, we find that

$$H_\tau(\sigma)\mathrm{e}^{-\frac{1}{4}\lambda^2 g_r t} \leq \prod_{j=1}^{|\sigma|} h_\tau(v_j - u_j)\mathrm{e}^{-\frac{1}{8}\lambda^2 g_r |v_j - u_j|} \tag{9.41}$$

We estimate the LHS of (9.38), with $\mathrm{e}^{-zt}\mathrm{e}^{(c'(\gamma,\lambda)-\frac{1}{4}\lambda^2 g_r)t}$ replaced by 1, by invoking (D.2) in Lemma D.1, with

$$k(t) := c(\gamma)\mathrm{e}^{-\frac{1}{8}\lambda^2 g_r t} h_\tau(t) \qquad \text{and} \qquad a := 0 \tag{9.42}$$

Indeed, by using $\|h_\tau\|_1 = o(\lambda^2)$ and the exponential decay $\mathrm{e}^{-\frac{1}{8}\lambda^2 g_r t}$, we obtain that

$$\|k\|_1 \quad = \quad c(\gamma)o(\lambda^2) \tag{9.43}$$

$$\|tk\|_1 \quad = \quad c(\gamma)o(|\lambda|^0), \qquad \text{as } \lambda \searrow 0 \tag{9.44}$$

Therefore, the bound (D.2) yields (9.38). □

9.4 List of important parameters

We list some constants and functions that we use throughout the paper. We start with the different decay rates; in the third column we indicate where the constant appears first. By "full model", we mean: "the model without cutoff".

g_{R}	bare reservoir correlation fct. (for subluminal speed)	Lemma 5.1
$\frac{1}{\tau} = \lvert\lambda\rvert^{1/2}$	bare reservoir correlation fct. in the cut-off model	Section 5.3
$\lambda^2 g_r$	renormalized joint $\mathrm{S}-\mathrm{R}$ correlation function	Lemma 9.2
$\lambda^2 g_{rw}$	Markov semigroup	Proposition 4.2
$\lambda^2 g_c$	cut-off model	Lemma 6.2
$\lambda^2 g_{ex}$	excitations in the full model	Lemma 9.1
$\lambda^2 g$	full model	Theorem 3.2

Additionally, the rates g_{rw}, g_c, g come with a superscript $low, high$ indicating that the gap refers to small, large fibers p, respectively.

The following constants restrict the values of complex deformation parameters, in particular the parameter κ in $\mathcal{J}_\kappa$, as defined in 2.59.

δ_ε	particle dispersion law	Assumption 2.1
δ_{R}	reservoir dispersion law	Assumption 2.3
δ	full model	Theorem 3.3
δ_{rw}	full Markov semigroup	Proposition 4.2
δ_{ex}	excitations in the full model	Lemma 6.5
δ_r	renormalized $\mathrm{S}-\mathrm{R}$ correlation fct.	Lemma 9.2

The following functions of γ, λ appear as blowup-rates in exponential bounds.

$r_{rw}(\gamma,\lambda)$	Markov semigroup	Lemma 8.1
$r_\varepsilon(\gamma,\lambda)$	$\tilde{\mathcal{U}}_t^\gamma$	Section 8.3
$r_\tau(\gamma,\lambda)$	$\tilde{\mathcal{Z}}_t^{\tau\gamma}$	Lemma 6.2

In the final Sections 8 and 9, these rates are represented by the generic constants $c(\gamma,\lambda), c'(\gamma,\lambda)$, introduced in Section 8.1.

A Appendix: The reservoir correlation function

In this appendix, we study the reservoir correlation function $\psi(x,t)$ and we prove Lemmata 5.1 and 5.2. Recall the definition of the "effective squared form factor" $\hat{\psi}$ in (2.27). It is related to $\psi(x,t)$ by, see (5.3),

$$\psi(x,t) = \int_{\mathbb{R}} \mathrm{d}\omega \int_{\mathbb{S}^{d-1}} \mathrm{d}s\, \hat{\psi}(\omega)\mathrm{e}^{\mathrm{i}t\omega}\mathrm{e}^{\mathrm{i}\omega s\cdot x} \tag{A.1}$$

From this expression, one understands that $\psi(x,t)$ cannot have exponential decay in t, uniformly in x. One also sees that, for x fixed, there is exponential decay provided that $\hat{\psi}(\cdot)$ is analytic in a strip around $\mathbb{R}$.

Let $q(\cdot)$ be the Fourier transform of $\hat{\psi}$, then

$$\psi(x,t) = \int_{\mathbb{S}^{d-1}} \mathrm{d}s\, q(t+s\cdot x) \tag{A.2}$$

By Assumption 2.3, there is a $\delta_{\mathrm{R}} > 0$ such that $q(t)$ decays as $C\mathrm{e}^{-\delta_{\mathrm{R}}|t|}$. Choosing $v^* = \frac{1}{2}$, we obtain, for $|x|/|t| \leq v^*$,

$$|\psi(x,t)| \leq \mathrm{e}^{-g_{\mathrm{R}}|t|}C, \qquad \text{with } g_{\mathrm{R}} := \frac{2}{3}\delta_{\mathrm{R}} \tag{A.3}$$

which proves Lemma 5.1. From now on, we assume that $|x|/|t| > v^*$.

We remark that (A.2) can be rewritten, after an explicit calculation, as

$$\psi(x,t) = \int_{-1}^{1} \mathrm{d}\eta\, q(t+\eta|x|)a(\eta), \qquad a(\eta) := Volume(\mathbb{S}^{d-2})\left(1-\eta^2\right)^{\frac{d-3}{2}} \tag{A.4}$$

By Assumption 2.3, in particular the condition $\hat{\psi}(0) = 0$ and the analyticity of $\hat{\psi}$, we deduce that $\frac{\hat{\psi}(\omega)}{\omega}$ is analytic in a strip around $\mathbb{R}$, as well. Its Fourier transform, Q, is an exponentially decaying C^1-function (since $\frac{\hat{\psi}(\omega)}{\omega} \in L^1$) whose derivative equals q.

Hence, by partial integration, the fact that $a(\eta)\big|_{-1} = a(\eta)\big|_{1} = 0$ for $d > 3$, and the change of variables $\zeta = |x|\eta$, we obtain

$$\psi(x,t) = -\frac{1}{|x|^{3/2}}\int_{-|x|}^{|x|} \mathrm{d}\zeta\, Q(t+\zeta)\,\frac{1}{|x|^{1/2}}a'(\frac{\zeta}{|x|}), \qquad Q' = q \tag{A.5}$$

Here, Q' and a' stand for the derivatives of Q and a. We evaluate this integral by splitting it into the regions

$$-|x| \leq \zeta \leq -|x|+1, \qquad -|x|+1 \leq \zeta \leq |x|-1, \qquad |x|-1 \leq \zeta \leq |x| \tag{A.6}$$

In the second region, we dominate the integral (A.5) by $\|Q\|_1 \times \|\frac{1}{|x|^{1/2}}a'(\frac{\zeta}{|x|})\|_\infty$ (we assume here that $|x| \geq 1$, otherwise the decay in t has been proven above). In the first and third region, we dominate the integral (A.5) by $\|Q\|_\infty \times \|\frac{1}{|x|^{1/2}}a'(\frac{\zeta}{|x|})\|_1$. Using the explicit form of a' and the fact that $|x|/|t| > v^*$, we conclude

$$\sup_x |\psi(x,t)| \leq C(1+|t|)^{-3/2}, \qquad \text{for } d \geq 4 \tag{A.7}$$

which implies Lemma 5.2. Obviously, dispersive estimates (A.7) can be derived in much greater generality, see e.g. [18].

B Appendix: Spectral perturbation theory

Let $\epsilon \in \mathbb{R}$ be a small parameter and consider a continuous function $\mathbb{R}^+ \ni t \mapsto V(t,\epsilon)$, taking values in a Banach space, and such that

$$\sup_{t\geq 0} \mathrm{e}^{-tm}\|V(t,\epsilon)\| < \infty, \qquad \text{for some } m > 0. \tag{B.1}$$

The Laplace transform

$$A(z,\epsilon) := \int_{\mathbb{R}^+} \mathrm{d}t \mathrm{e}^{-tz} V(t,\epsilon) \tag{B.2}$$

is well-defined for $\operatorname{Re} z > m$ and it follows (by the inverse Laplace transform) that

$$V(t,\epsilon) = \frac{1}{2\pi \mathrm{i}} \int\limits_{\Gamma^{\rightarrow}} \mathrm{d}z\, \mathrm{e}^{zt} A(z,\epsilon), \qquad \text{with } \Gamma^{\rightarrow} := m' + \mathrm{i}\mathbb{R} \text{ for any } m' > m \tag{B.3}$$

where the integral is in the sense of improper Riemann integrals.

We will state assumptions that allow to continue $A(z,\epsilon)$ downwards in the complex plane, i.e., to $\operatorname{Re} z \leq m$ and to obtain bounds on $V(t,\epsilon)$.

Lemma B.1. *For $\operatorname{Re} z$ large enough, let*

$$A(z,\epsilon) := (z - \mathrm{i}B - A_1(z,\epsilon))^{-1} \tag{B.4}$$

and assume the following conditions.

1) *B is bounded and its spectrum consists of finitely many semisimple eigenvalues on the real axis, that is*

$$B = \sum_{b \in \mathrm{sp} B} b 1_b(B), \tag{B.5}$$

where $1_b(B)$ is the spectral projection corresponding to the eigenvalue b. For concreteness, we assume that $0 \in \mathrm{sp} B$.

2) *For ϵ small enough, the operator-valued function $z \mapsto A_1(z,\epsilon)$ is analytic in the domain $\operatorname{Re} z > -\epsilon g_A$ and*

$$\sup_{\operatorname{Re} z > -\epsilon g_A} \|A_1(z,\epsilon)\| = O(\epsilon), \tag{B.6}$$

$$\sup_{\operatorname{Re} z > -\epsilon g_A} \|\frac{\partial}{\partial z} A_1(z,\epsilon)\| = o(|\epsilon|^0), \qquad \epsilon \searrow 0 \tag{B.7}$$

3) *There are bounded operators N_b, for $b \in \mathrm{sp} B$, acting on the spectral subspaces $\mathrm{Ran} 1_b(B)$ and such that, for all $b \in \mathrm{sp} B$,*

$$\epsilon N_b - 1_b(B) A_1(\mathrm{i}b,\epsilon) 1_b(B) = o(\epsilon), \qquad \epsilon \searrow 0. \tag{B.8}$$

Consider the operator

$$N := \bigoplus_{b \in \mathrm{sp} B} N_b, \qquad \textit{with } [B,N] = 0, \tag{B.9}$$

and assume that N has a simple eigenvalue f_N such that

$$\mathrm{sp} N = \{f_N\} \cup \Omega_N \qquad \textit{and} \qquad \sup \operatorname{Re} \Omega_N \leq -g_N \tag{B.10}$$

for some gap $g_N > 0$. We also require that

$$\operatorname{Re} f_N > -g_N, \qquad \operatorname{Re} f_N > -g_A \tag{B.11}$$

The eigenvalue f_N is necessarily an eigenvalue of N_b for some $b \in \mathrm{sp} B$. For concreteness (and to match with our applications), we assume that it is an eigenvalue of N_0

Then, there is an $\epsilon_0 > 0$ such that, for $|\epsilon| \leq \epsilon_0$, there is a number $f(\epsilon)$, a rank-one operator $P(\epsilon)$, bounded operators $R(t,\epsilon)$ and a decay rate $g > 0$, such that

$$V(t,\epsilon) = P(\epsilon) \mathrm{e}^{f(\epsilon)t} + R(t,\epsilon) \mathrm{e}^{-\epsilon g t} \tag{B.12}$$

with

$$f(\epsilon) - \epsilon f_N = o(\epsilon) \tag{B.13}$$

$$\|P(\epsilon) - 1_{f_N}(N)\| = o(|\epsilon|^0) \tag{B.14}$$

$$\sup_{t \in \mathbb{R}^+} \|R(t,\epsilon)\| = O(|\epsilon|^0), \qquad \textit{as } |\epsilon| \searrow 0 \tag{B.15}$$

with $1_{f_N}(N)$ the spectral projection of N associated to the eigenvalue f_N. The decay rate g can be chosen arbitrarily close to $\min\{g_N, g_A\}$ by making ϵ_0 small enough. In particular, one can choose g and ϵ_0 such that $\mathrm{Re}\, f(\epsilon) > -\epsilon g$ for all $|\epsilon| \leq \epsilon_0$.

If, in addition N and A_1 depend analytically on a parameter α in a complex domain $\mathcal{D} \subset \mathbb{C}$, such that (B.6)-(B.7)-(B.8)-(B.10)-(B.11) *hold uniformly in $\alpha \in \mathcal{D}$, then* (B.12) *holds with f, P and R analytic in α and the estimates* (B.13)-(B.14)-(B.15) *are satisfied uniformly in $\alpha \in \mathcal{D}$.*

Lemma B.1 follows in a straightforward way from spectral perturbation theory of discrete spectra. For completeness, we give a proof below, using freely some well-known results that can be found in, e.g., [26].

Lemma B.2. *The singular points of $A(z, \epsilon)$ in the domain $\mathrm{Re}\, z \geq -\epsilon g_A$ lie within a distance of $o(\epsilon)$ of the spectrum of $\mathrm{i}B + \epsilon N$ (provided that there are any singular points at all).*

Proof. Standard perturbation theory implies that the spectrum of the operator

$$\mathrm{i}B + A_1(z, \epsilon), \qquad \text{for } \mathrm{Re}\, z \geq -\epsilon g_A \tag{B.16}$$

lies at a distance $O(\epsilon)$ from the spectrum of $\mathrm{i}B$. Here and in what follows, the estimates in powers of ϵ are uniform for $\mathrm{Re}\, z \geq -\epsilon g_A$. Let $1_b^0 \equiv 1_b(B)$ be the spectral projections of B on the eigenvalue b. As long as ϵ is small enough, there is an invertible operator $U \equiv U(\epsilon, z)$ satisfying $\|U - 1\| = O(\epsilon)$ and such that the projections

$$1_b := U 1_b^0 U^{-1}, \qquad b \in \mathrm{sp} B \tag{B.17}$$

are spectral projections of the operator (B.16) associated to the spectral patch originating from the eigenvalue b at $\epsilon = 0$. It follows that the spectral problem for (B.16) is equivalent to the spectral problem for

$$\sum_b U^{-1} 1_b \left(\mathrm{i}B + A_1(z, \epsilon)\right) 1_b U = \sum_b \left(\mathrm{i}b 1_b^0 + \epsilon N_b + A_{ex,b}(z, \epsilon)\right) \tag{B.18}$$

where

$$\begin{array}{rcll} A_{ex,b}(z,\epsilon) & := & 1_b^0 U^{-1} (\mathrm{i}B) U 1_b^0 - \mathrm{i}b 1_b^0, & (O(\epsilon^2)) \\ & + & \epsilon 1_b^0 U^{-1} N_b U 1_b^0 - \epsilon N_b, & (O(\epsilon^2)) \\ & + & 1_b^0 U^{-1} \left(A_1(\mathrm{i}b, \epsilon) - \epsilon N_b\right) U 1_b^0, & (o(\epsilon)) \\ & + & 1_b^0 U^{-1} \left(A_1(z, \epsilon) - A_1(\mathrm{i}b, \epsilon)\right) U 1_b^0, & (|z - \mathrm{i}b| o(|\epsilon|^0)) \end{array} \tag{B.19}$$

The estimates in powers of ϵ are obtained by using $U - 1 = O(\epsilon)$, the property $1_b U = U 1_b^0$ and the bounds (B.6)-(B.7)-(B.8). When z is chosen at a distance $O(\epsilon)$ from $\mathrm{i}b$, then all terms in (B.19) are $o(\epsilon)$. The claim of Lemma B.2 now follows by simple perturbation theory applied to the RHS of (B.18).

□

Lemma B.3. *The function $A(z)$ has exactly one singularity at a distance $o(\epsilon)$ from ϵf_N. This singularity is called $f \equiv f(\epsilon)$. The corresponding residue $P \equiv P(\epsilon)$ is a rank-one operator satisfying*

$$\|P - 1_N(f_N)\| = o(|\epsilon|^0), \qquad \epsilon \searrow 0 \tag{B.20}$$

Proof. By Lemma B.2, there can be at most one singularity. We prove below that there is at least one. By the reasoning in the proof of Lemma B.2 and the fact that the eigenvector corresponding to f_N belongs to $\mathrm{Ran} 1_{b=0}^0$ (see condition 3) of Lemma B.1), it suffices to study the singularities of the function

$$z \mapsto \left(z - \epsilon N_0 + A_{ex,0}(z, \epsilon)\right)^{-1} \tag{B.21}$$

Let the contour $\Gamma^f \equiv \Gamma^f(\epsilon)$ be a circle with center ϵf_N and radius ϵr for some $r > 0$. Clearly, for r small enough, the entire spectrum of ϵN_0 lies outside the contour Γ^f, except for the eigenvalue ϵf_N. The contour integral of $(z - \epsilon N)^{-1}$

along Γ^f equals the spectral projection corresponding to f_N. We estimate

$$\oint_{\Gamma^f} \mathrm{d}z \, \left[(z - \epsilon N_0 - A_{ex,0}(z,\epsilon))^{-1} - (z - \epsilon N_0)^{-1} \right] \tag{B.22}$$

$$= \oint_{\Gamma^f} \mathrm{d}z \, (z - \epsilon N_0)^{-1} A_{ex,0}(z,\epsilon)(z - \epsilon N_0 - A_{ex,0}(z,\epsilon))^{-1} \tag{B.23}$$

$$= \oint_{\Gamma^f} \mathrm{d}z \, (\epsilon^{-2} c(r))^2 o(\epsilon), \qquad \text{as } \epsilon \searrow 0 \text{ with } c(r) := \sup_{|z - f_N| = r} \|(z - N_0)^{-1}\|, \tag{B.24}$$

The last estimate holds in norm and it follows from the bound $\|A_{ex,0}(z,\epsilon)\| = o(\epsilon)$, see (B.19). The expression (B.24) is $o(1)$, as $\epsilon \searrow 0$, since the circumference of the contour Γ^f is $2\pi r \epsilon$. From the fact that the contour integral of (B.21) does not vanish, we conclude that $A(z)$ has at least one singularity inside Γ^f.

The claim about the residue is most easily seen in an abstract setting. Let $F(z)$ be a Banach-space valued analytic function in some open domain containing 0, and such that $0 \in \mathrm{sp} F(0)$ is an isolated eigenvalue. We have the Taylor expansion

$$F(z) = \sum_{n \geq 0} \frac{z^n}{n!} F_n, \qquad F_n := F^{(n)}(0), \qquad 0 \in \mathrm{sp} F_0 \tag{B.25}$$

If $\|F_1 - 1\|$ is small enough, then also $F_1^{-1} F_0$ has 0 as an isolated eigenvalue. We denote the corresponding spectral projection by $1_0(F_1^{-1} F_0)$ and we calculate

$$\mathrm{Res}(F(z)^{-1}) = \mathrm{Res}(F_0 + z F_1)^{-1} = \left(\mathrm{Res}(F_1^{-1} F_0 + z)^{-1} \right) F_1^{-1} = 1_0(F_1^{-1} F_0) F_1^{-1}. \tag{B.26}$$

The last expression is clearly a rank-one operator. In the case at hand, $F_1^{-1} = 1 + o(|\epsilon|^0)$, as $\epsilon \searrow 0$, which yields (B.20). □

We proceed to the proof of Lemma B.1.

First, we choose the rate g such that $f_N < g < \min\{g_A, g_N\}$ and we fix the contours Γ^f and $\Gamma_{\rightarrow}$ (see also Figure 15);

- The contour Γ^f is as described in Lemma B.3, with $r < |g - f_N|$. In particular, for small ϵ, it encircles the point f but no other singular points of $A(z)$.
- The contour $\Gamma_{\rightarrow}$ is given by $\Gamma_{\rightarrow} := -\epsilon g + \mathrm{i}\mathbb{R}$.

By Lemma B.2, we know that for small ϵ, there are no singularities of $A(z)$ in the region $\mathrm{Re}\, z > -\epsilon g$ except for the point $z \equiv f$. Hence, we can deform contours as follows

$$V(t,\epsilon) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\rightarrow}} \mathrm{d}z \, \mathrm{e}^{zt} A(z,\epsilon) \tag{B.27}$$

$$= \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma^f} \mathrm{d}z \, \mathrm{e}^{zt} A(z,\epsilon) + \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\rightarrow}} \mathrm{d}z \, \mathrm{e}^{zt} A(z,\epsilon) \tag{B.28}$$

The first term in (B.28) yields $\mathrm{e}^{tf} P$. The second term of (B.28) is split as follows

$$\int_{\Gamma_{\rightarrow}} \mathrm{d}z \, \mathrm{e}^{zt} A(z,\epsilon) \tag{B.29}$$

$$= \int_{\Gamma_{\rightarrow}} \mathrm{d}z \, \mathrm{e}^{zt} (z - \mathrm{i}B - \epsilon N)^{-1} \tag{B.30}$$

$$+ \int_{\Gamma_{\rightarrow}} \mathrm{d}z \, \mathrm{e}^{zt} (z - \mathrm{i}B - \epsilon N)^{-1} (A_1(z,\epsilon) - \epsilon N) A(z,\epsilon) \tag{B.31}$$

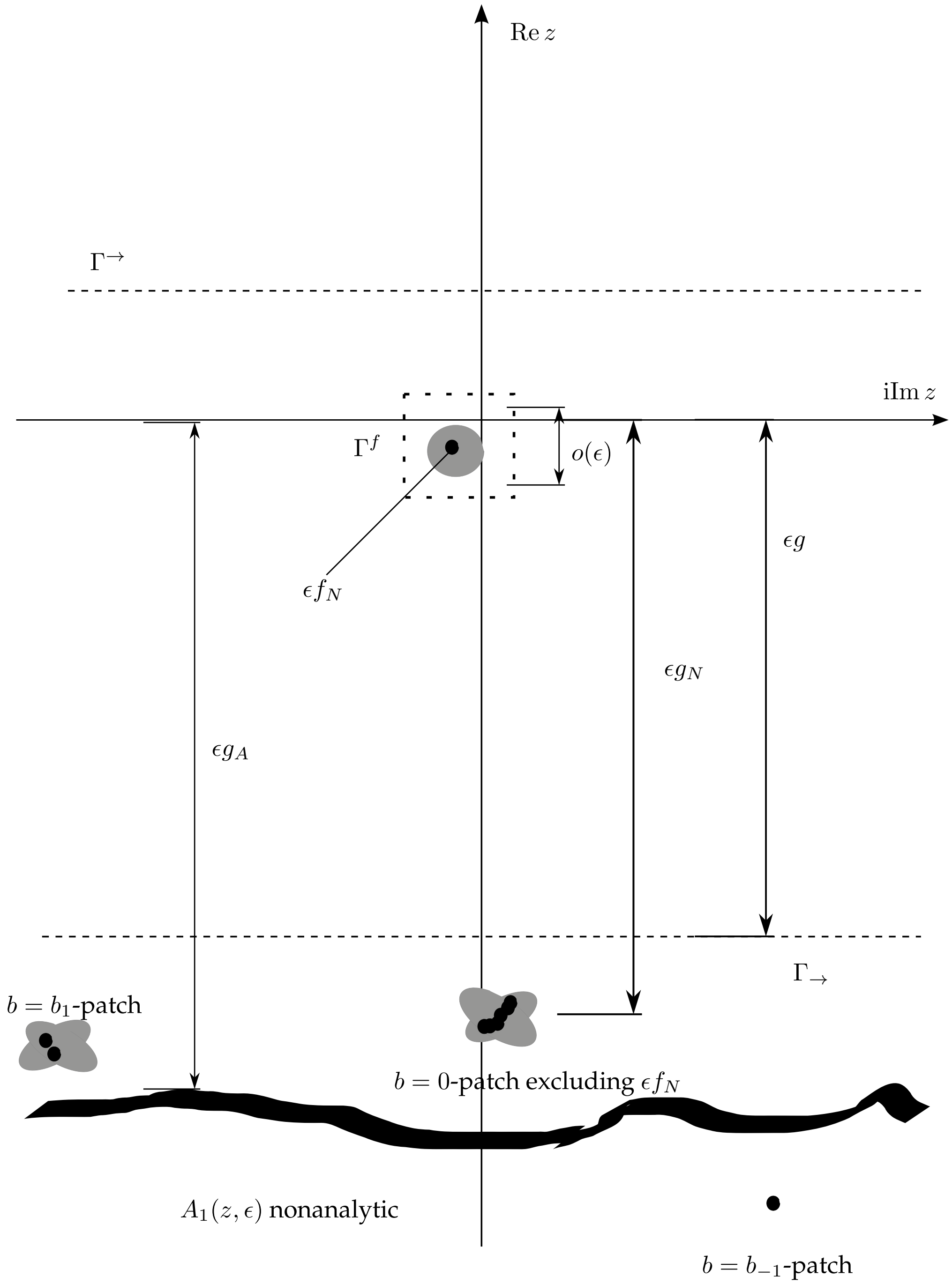


Figure 15: The (rotated) complex plane. The black dots indicate the spectrum of $\mathrm{i}B + \epsilon N$ (which need not be discrete). The upper dot is the eigenvalue ϵf_N. In the picture, we have assumed that the spectrum of B consists of 3 semisimple eigenvalues: $0, b_1, b_{-1}$. The gray patches contain the possible singularities of the function $A(z)$ above the irregular black line. These singularities lie at $o(\epsilon)$ from the spectrum of $\mathrm{i}B+\epsilon N$. Below the irregular black line, i.e., in the region $\mathrm{Re}\, z < -\epsilon g_A$, we have no control since $A_(z, \epsilon)$ ceases to be analytic in that region (hence we have also not drawn a patch around b_{-1}). The integration contours $\Gamma^{\rightarrow}, \Gamma_{\rightarrow}$ and Γ^{f} are drawn as dashed lines.

The term (B.30) equals

$$\mathrm{e}^{t\left(\mathrm{i}B+\epsilon 1_{\Omega_N}(N)N\right)} = O(\mathrm{e}^{-\epsilon g_N t}), \qquad t \nearrow \infty \tag{B.32}$$

since the contour $\Gamma_{\rightarrow}$ can be closed in the lower half-plane to enclose the spectrum of $\mathrm{i}B+\epsilon N$ minus the eigenvalue ϵf_N, i.e., the set $\epsilon\Omega_N$.

The integrand of (B.31) decays as $|z|^{-2}$ for $z \nearrow \infty$, since for a bounded operator M

$$\|(z-M)^{-1}\| = O(\frac{1}{|z|}), \qquad |z| \nearrow \infty \tag{B.33}$$

Using that $A_1(z,\epsilon) = O(\epsilon)$, it is now easy to establish that the integral in (B.31) is $O(1)$, as $\epsilon \searrow 0$. One extracts $\mathrm{e}^{t\mathrm{Re}\, z}$ from the integration (B.31) to get the bound $O(\mathrm{e}^{-\epsilon g t})$. Together with (B.32), this proves Lemma B.1.

C Appendix: Construction and analysis of the Lindblad generator $\mathcal{M}$

The operator $\mathcal{M}$ was introduced at the beginning of Section 4. We provide a more explicit construction and we prove Propositions 4.1 and 4.2.

C.1 Construction of $\mathcal{M}$

First, we note that by using the notions introduced in Section 5.2, the operator $\mathcal{L}(z)$, defined in Section 4.1, can be expressed as

$$\mathcal{L}(z) = \int_{\mathbb{R}^+} \mathrm{d}t\, \mathrm{e}^{-tz} \sum_{x_1,x_2,l_1,l_2} \psi^{\#}(x_2-x_1,t)\, \mathcal{I}_{x_2,l_2} \mathrm{e}^{-\mathrm{i}\mathrm{ad}(Y)t} \mathcal{I}_{x_1,l_1} \tag{C.1}$$

where $\psi^{\#}$ equals ψ or $\overline{\psi}$, depending on l_1,l_2, according to the rules in (5.10). In words, $\lambda^2\mathcal{L}(z)$ contains the terms of order λ^2 in the Lie-Schwinger series of Lemma 2.5.

Next, we define some auxiliary objects.

$$\Upsilon \;:=\; \mathrm{Im} \sum_{a\in \mathrm{sp}(\mathrm{ad}(Y))} W_a W_a^* \int_{\mathbb{R}^+} \mathrm{d}t\, \psi(0,t)\mathrm{e}^{\mathrm{i}at} \tag{C.2}$$

$$\Psi(\rho) \;:=\; \sum_{x_L,x_R\in\mathbb{Z}^d} \sum_{a\in \mathrm{sp}(\mathrm{ad}(Y))} \left(\int_{\mathbb{R}} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}at}\psi(x_L-x_R,t)\right) \times\, (1_{x_L}\otimes W_a)\rho(1_{x_R}\otimes W_a)^* \tag{C.3}$$

The operator $\Upsilon = \Upsilon^* \in \mathscr{B}(\mathscr{S})$ was already referred to in Section 4. From the above expression and the definition of W_a in (2.30), we check immediately that $[Y,\Upsilon]=0$. Further, we can rewrite (C.3) as

$$\Psi(\rho) = 2\pi \sum_{a\in \mathrm{sp}(\mathrm{ad}(Y))} \int_{\mathbb{S}^{d-1}} \mathrm{d}s\, \hat{\psi}(a) V(s,a)\rho V^*(s,a) \tag{C.4}$$

with $\hat{\psi}(\cdot)$ as in (2.27) and (5.3), and

$$V(s,a) := \sum_{x\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}as\cdot x} 1_x \otimes W_a \tag{C.5}$$

The expression (C.4) is essentially the Kraus decomposition of Ψ, see [1], and hence it shows that Ψ is a completely positive map. This means in particular that for $\rho\geq 0$, $\|\Psi(\rho)\|_1 = \mathrm{Tr}\,\Psi(\rho)$, and hence, by using (C.3),

$$\|\Psi\rho\|_1 = \sum_x \mathrm{Tr}_{\mathscr{S}}\left((\Psi\rho)(x,x)\right) \leq \left(\int \mathrm{d}t|\psi(0,t)|\|W\|^2\right)\|\rho\|_1 \tag{C.6}$$

where the finiteness of the factor between brackets on the RHS is implied by Lemma 5.2. Since any trace class operator can be written as a linear combination of four positive trace class operators, it follows that Ψ is bounded on $\mathscr{B}_1(\mathscr{H}_{\mathrm{S}})$ and by a similar calculation, one can check that Ψ is also bounded on $\mathscr{B}_2(\mathscr{H}_{\mathrm{S}})$.

We are now ready to verify that

$$\mathcal{M}(\rho) = -\mathrm{i}[\varepsilon(P) + \Upsilon, \rho] + \Psi(\rho) - \frac{1}{2}(\Psi^*(1)\rho + \rho\Psi^*(1)). \tag{C.7}$$

Indeed, this is checked most conveniently starting from (4.4) and employing (C.1). The terms with $l_1 \neq l_2$ give rise to $\Psi(\rho)$, while the terms with $l_1 = l_2$ give rise to $-\mathrm{i}[\Upsilon, \rho]$ and $-\frac{1}{2}(\Psi^*(1)\rho + \rho\Psi^*(1))$. Moreover, by the boundedness and complete positivity of Ψ and the representation (C.7), it follows that $\mathcal{M}$ is of Lindblad type, see e.g. [1]. Starting from (C.4), one can derive the momentum space representation of $\mathcal{M}$ given in Section 4.2. For example, by expressing $V(s,a)$ in momentum representation, one obtains

$$\Psi(\rho)(k_L, k_R) = 2\pi \sum_{a \in \mathrm{sp}(\mathrm{ad}(Y))} \int_{\mathbb{S}^{d-1}} \mathrm{d}s\, \hat{\psi}(a) W_a \rho(k_L + sa, k_R + sa) W_a^* \tag{C.8}$$

which gives rise to the first term of (4.16).

C.1.1 Proof of Proposition 4.1

By the integrability in time of the correlation function $\psi(x,t)$, as stated in Lemma 5.2, the expression (C.1) implies immediately that $\mathcal{L}(z)$ can be continued continuously to $z \in \mathbb{R}$. This proves (4.5). The boundedness of $\mathcal{M}$ on $\mathscr{B}_2(\mathscr{H}_{\mathrm{S}})$ and $\mathscr{B}_1(\mathscr{H}_{\mathrm{S}})$ follows from the boundedness of Ψ, which was explained above. The complete positivity of the map Ψ and the canonical form (C.7) imply that $\mathcal{M}$ is a Lindblad generator, see e.g. [1]. Consequently, $-\mathrm{iad}(Y) + \lambda^2\mathcal{M}$ is also a Lindblad generator and the semigroup Λ_t is positivity-preserving and trace-preserving. To check (4.6), we note that

$$\mathcal{J}_\kappa \mathcal{M} \mathcal{J}_{-\kappa} - \mathcal{M} = -\mathrm{i}\left[\mathcal{J}_\kappa \mathrm{ad}(\varepsilon(P))\mathcal{J}_{-\kappa} - \mathrm{ad}(\varepsilon(P))\right] \tag{C.9}$$

and hence (4.6) follows immediately from Assumption 2.1.

C.2 Spectral analysis and proof of Proposition 4.2

The claims of Proposition 4.2 require a spectral analysis which we present now. We recall the decomposition

$$\mathcal{M} = \int_{\mathbb{T}^d}^{\oplus} \mathrm{d}p\, \mathcal{M}_p = \int_{\mathbb{T}^d}^{\oplus} \mathrm{d}p \underset{a \in \mathrm{sp}(\mathrm{ad}(Y))}{\oplus} \mathcal{M}_{p,a} \tag{C.10}$$

and we keep in mind that Proposition 4.2 treats $\mathcal{M}_p$ as an operator on the Hilbert space $\mathscr{G} \sim L^2(\mathbb{T}^d, \mathscr{B}_2(\mathscr{S}))$.

C.2.1 Explicit representation of $\mathcal{M}_{p,0}$

By exploiting the nondegeneracy condition in Assumption 2.4, we can identify $\mathcal{M}_{p,a=0}$, for each p with an operator on $L^2(\mathbb{T}^d \times \mathrm{sp}Y)$. This was explained in Section 4.2. We introduce explicitly *gain*, *loss* and *kinetic* operators operators; G, L and K_p, acting on $L^2(\mathbb{T}^d \times \mathrm{sp}Y)$, by

$$G\varphi(k,e) := \sum_{e' \in \mathrm{sp}Y} \int_{\mathbb{T}^d} \mathrm{d}k' r(k', e'; k, e)\varphi(k', e') \tag{C.11}$$

$$L\varphi(k,e) := -\sum_{e' \in \mathrm{sp}Y} \int_{\mathbb{T}^d} \mathrm{d}k' r(k, e; k', e')\varphi(k, e) \tag{C.12}$$

$$K_p\varphi(k,e) := \mathrm{i}(\varepsilon(k+\frac{p}{2}) - \varepsilon(k-\frac{p}{2}))\varphi(k,e), \qquad \varphi \in L^2(\mathbb{T}^d \times \mathrm{sp}Y) \tag{C.13}$$

The kinetic operator K_p models the free flight of the particle between collisions. The operators L and K_p act by multiplication in the variables k, e. The expression for $\mathcal{M}_{p,0}$, given in given in Section 4.2 (in particular in (4.15)), can be rewritten as

$$\mathcal{M}_{p,0} = G + L + K_p \tag{C.14}$$

We define the similarity transformation

$$A \mapsto \hat{A} := \mathrm{e}^{\frac{1}{2}\beta Y} A \mathrm{e}^{-\frac{1}{2}\beta Y}, \qquad \text{for } A \in \mathscr{B}(L^2(\mathbb{T}^d \times \mathrm{sp}Y)) \tag{C.15}$$

where we have slightly abused the notation by writing Y to denote a multiplication operator on $\mathrm{sp}Y$, i.e., $Y\varphi(k,e) = e\varphi(k,e)$. Since L and K_p act by multiplication, we have $\hat{L} = L$ and $\hat{K}_p = K_p$. The usefulness of this similarity transformation resides in the fact that $\hat{G}$, and hence also $\hat{\mathcal{M}}_{0,0}$, are self-adjoint on $L^2(\mathbb{T}^d \times \mathrm{sp}Y)$.

C.2.2 Explicit representation of $\mathcal{M}_{p,a\neq 0}$

To write an explicit expression for $\mathcal{M}_{p,a\neq 0}$, we first define the operators (acting on $\mathscr{B}(\mathscr{S})$)

$$\mathrm{ad}_a(\Upsilon) = 1_a(\mathrm{ad}(Y))\mathrm{ad}(\Upsilon)1_a(\mathrm{ad}(Y)), \qquad a \in \mathrm{sp}(\mathrm{ad}(Y)) \tag{C.16}$$

which satisfy $\mathrm{ad}_0(\Upsilon) = 0$ and $\mathrm{ad}(\Upsilon) = \oplus_a \mathrm{ad}_a(\Upsilon)$ since $[\Upsilon, Y] = 0$. Due to the nondegeneracy condition in Assumption 2.4, both $1_a(\mathrm{ad}(Y))$ and $\mathrm{ad}_a(\Upsilon)$ are rank-one operators and we identify $\mathrm{ad}_a(\Upsilon)$ with a number Υ_a, such that $\mathrm{ad}_a(\Upsilon) = \Upsilon_a 1_a(\mathrm{ad}(Y))$. In fact, as already remarked in Section 4.2, the operator $\mathcal{M}_{p,a\neq 0}$ itself acts as the rank-one operator $1_a(\mathrm{ad}(Y))$ on $\mathscr{B}(\mathscr{S})$ and hence we identify it with an operator on $L^2(\mathbb{T}^d)$ (which is also called $\mathcal{M}_{p,a\neq 0}$ here):

$$\mathcal{M}_{p,a\neq 0}\varphi(k) = -\mathrm{i}(\Upsilon_a + \varepsilon(k + \frac{p}{2}) - \varepsilon(k - \frac{p}{2}))\varphi(k) - \frac{1}{2}\left(j(k,e) + j(k,e')\right)\varphi(k), \qquad \varphi \in L^2(\mathbb{T}^d) \tag{C.17}$$

where $j(k,e)$ are the escape rates introduced in (4.27) and e, e' are determined by $a = e - e'$. To check (C.17), one starts from (4.15) and one uses

- The fact that $r(k', e'; k, e)$ vanishes for $e' = e$, as remarked following (4.17).
- The definition of the matrices W_a in (2.30) and the escape rates $j(\cdot,\cdot)$ in (4.27).
- The definition $\varphi(k) \equiv \langle e, \xi(k)e' \rangle_{\mathscr{S}}$ in (4.23).

In particular, the last term on the RHS of (C.17) appears because

$$\int_{\mathbb{T}^d} \mathrm{d}k' r_{e'-e}(k,k') \Big\langle e', (W_a W_a^* \xi(k) + \xi(k) W_a W_a^*)\, e \Big\rangle_{\mathscr{S}} = \left(j(k,e) + j(k,e')\right)\varphi(k) \tag{C.18}$$

Hence, $\mathcal{M}_{p,a\neq 0}$ acts by multiplication in the variable k.

C.2.3 Analysis of $\mathcal{M}_{0,0}$

We already established that $\mathcal{M}_{0,0}$ is a bounded Markov generator on $L^1(\mathbb{T}^d \times \mathrm{sp}Y)$. This implies that

$$\mathrm{Re}\,\mathrm{sp}_{L^1}\mathcal{M}_{0,0} \leq 0. \tag{C.19}$$

The operator $\hat{\mathcal{M}}_{0,0}$ is not longer a Markov generator, but its spectrum is identical to $\hat{\mathcal{M}}_{0,0}$, since $\mathrm{e}^{\pm\frac{1}{2}\beta Y}$ is bounded and invertible. The *loss* operator L is a multiplication operator and its spectrum is found to be (see (4.27))

$$\mathrm{sp}(L) = -\left\{ j(e,k) \,\middle|\, e \in \mathrm{sp}Y, k \in \mathbb{T}^d \right\} \tag{C.20}$$

It is important to note that the escape rates $j(e,k)$ are bounded away from 0; this is a consequence of Assumption 2.4 and more concretely, of the fact that for each e, there is a e' such that (2.34) holds. Hence, we have

$$\mathrm{sp}(L) < 0 \tag{C.21}$$

Next, we argue that G is a compact operator on L^2. Indeed, for fixed e, e', the kernel $r(k', e'; k, e)$ depends only on $\Delta k \equiv k - k'$ and, hence, its Fourier transform acts on $l^2(\mathbb{Z}^d)$ by multiplication with the function

$$\mathbb{Z}^d \ni x \mapsto \int_{\mathbb{T}^d} \mathrm{d}(\Delta k) \mathrm{e}^{\mathrm{i}\Delta k \cdot x} r(0, e'; \Delta k, e) \tag{C.22}$$

From the explicit expression for $r(k', e'; k, e)$, one checks that the function (C.22) decays at infinity if the dimension $d > 1$ (recall that $d \geq 4$ by Assumption 2.2). Hence G is compact.

Given the compactness of G, Weyl's theorem ensures that the self-adjoint operators $\hat{\mathcal{M}}_{0,0}$ and $\hat{L}$ have the same essential spectrum.

By inspection, we check that $\hat{\mathcal{M}}_{0,0}$ has an eigenvalue 0, corresponding to the eigenvector $\hat{\varphi}^{eq}(k,e) \equiv \mathrm{e}^{-\frac{\beta}{2}e}$. Note that the corresponding right eigenvector of $\mathcal{M}_{0,0}$ is the (unnormalized) Gibbs state $\varphi^{eq}(k,e) \equiv \mathrm{e}^{-\beta e}$ and the corresponding left eigenvector is the constant function, since indeed

$$\hat{\varphi}^{eq} = \mathrm{e}^{\frac{\beta}{2}Y} \varphi^{eq}, \qquad \hat{\varphi}^{eq} = \mathrm{e}^{-\frac{\beta}{2}Y} 1_{\mathbb{T}^d \times \mathrm{sp}Y} \tag{C.23}$$

Since any eigenvalue of $\hat{\mathcal{M}}_{0,0}$ on L^2 has to be an eigenvalue of $\hat{\mathcal{M}}_{0,0}$ on L^1 (Note that $L^2(\mathbb{T}^d \times \mathrm{sp}Y) \subset L^1(\mathbb{T}^d \times \mathrm{sp}Y)$), the relation (C.19) implies that there are no eigenvalues with strictly positive real part.

We now exploit a Perron-Frobenius type of argument to argue that the eigenvalue 0 is simple and that it is the only eigenvalue on the real axis. See e.g. Theorem 13.3.6 in [8] for a version of the Perron-Frobenius theorem that establishes this in our case, provided that the semigroup $\mathrm{e}^{t\hat{\mathcal{M}}_{0,0}}$ is irreducible, i.e., that for any nonnegative functions $\varphi \in L^1(\mathbb{T}^d \times \mathrm{sp}Y)$, the inclusion

$$\mathrm{Supp}\left(\mathrm{e}^{t\hat{\mathcal{M}}_{0,0}}\varphi\right) \subset \mathrm{Supp}(\varphi), \qquad (\mathrm{Supp}\,\text{stands for 'support'}) \tag{C.24}$$

implies that either $\mathrm{Supp}(\varphi) = \mathbb{T}^d \times \mathrm{sp}Y$ or $\varphi = 0$. This irreducibility criterion is easily checked starting from Assumption 2.4, in particular its rephrasing in terms of a connected graph. Theorem 13.3.6 yields that the eigenvalue 1 of $\mathrm{e}^{t\hat{\mathcal{M}}_{0,0}}$ is simple, which implies that the eigenvalue 0 of $\hat{\mathcal{M}}_{0,0}$ is simple. To exclude purely imaginary eigenvalues $\mathrm{i}b$ of $\hat{\mathcal{M}}_{0,0}$, we apply this theorem for t such that $\mathrm{e}^{\mathrm{i}bt} = 1$.

C.2.4 Analysis of $\mathcal{M}_{p,0}$ and $\mathcal{M}_{p,a}$

We investigate the spectrum of $\hat{\mathcal{M}}_{p,0}$ as follows. By the same reasoning as in Section C.2.3, any spectrum with real part greater than (the negative number) $\sup \mathrm{sp} L$ consists of eigenvalues of finite multiplicity. Assume that $\hat{\mathcal{M}}_p$ has an eigenvalue m_p with (right) eigenvector) $\hat{\varphi}_p$. Then

$$\begin{aligned} \mathrm{Re}\, m_p \langle \hat{\varphi}_p, \hat{\varphi}_p \rangle &= \mathrm{Re}\, \langle \hat{\varphi}_p, \hat{\mathcal{M}}_{p,0} \hat{\varphi}_p \rangle && \text{(C.25)} \\ &= \mathrm{Re}\, \langle \hat{\varphi}_p, K_p \hat{\varphi}_p \rangle + \mathrm{Re}\, \langle \hat{\varphi}_p, \hat{\mathcal{M}}_{0,0} \hat{\varphi}_p \rangle && \text{(C.26)} \end{aligned}$$

The first term in (C.26) vanishes because the multiplication operator K_p is purely imaginary. The second term can only become positive if $\hat{\varphi} = \hat{\varphi}^{eq}$, with $\hat{\varphi}^{eq}$ the eigenvector of $\hat{\mathcal{M}}_{0,0}$ corresponding to the eigenvalue 0. This means that either the eigenvalue m_p has strictly negative real part, or the vector $\hat{\varphi}^{eq}$ is an eigenvector of $\hat{\mathcal{M}}_{0,0}$ with eigenvalue 0. In the latter case, $\hat{\varphi}^{eq}$ must also be an eigenvector of K_p with eigenvalue 0, which can only hold if $\varepsilon(k + \frac{p}{2}) - \varepsilon(k - \frac{p}{2}) = 0$ for all k. This is however excluded by the condition (2.11) in Assumption 2.1.

We conclude that for all $p \in \mathbb{T}^d \setminus \{0\}$, we have $\mathrm{Re}\, \mathrm{sp} \mathcal{M}_p < 0$. By compactness of $\mathbb{T}^d$ and the lower semicontinuity of the spectrum, we deduce hence that

$$\sup_{\mathbb{T}^d \setminus I_0} \mathrm{Re}\, \mathrm{sp} \mathcal{M}_{p,0} = c(I_0) < 0, \qquad \text{for any neighborhood } I_0 \text{ of } 0 \tag{C.27}$$

For $a \neq 0$, the operator $\mathcal{M}_{p,a\neq 0}$ is a multiplication operator in k and

$$\mathrm{Re}\, \mathrm{sp} \mathcal{M}_{p,a} \leq -\frac{1}{2} \inf_{k,e} j(k,e) < 0, \qquad \text{independently of}\, p \tag{C.28}$$

as follows by (C.17) and the fact that $j(k,e)$ is bounded away from 0.

C.2.5 Proof of Proposition 4.2

We summarize the results of Sections C.2.3 and C.2.4. For $a \neq 0$, the real part of the spectrum of the operators $\mathcal{M}_{p,a}$ is strictly negative, uniformly in p, see (C.28). The real part of the spectrum of $\mathcal{M}_{p,0}$ is strictly negative, uniformly in p except for a neighborhood of $0 \in \mathbb{T}^d$.

The operator $\mathcal{M}_{0,0}$ has a simple eigenvalue at 0 with corresponding eigenvector φ^{eq}, as defined in Section C.2.3. The rest of the spectrum of $\mathcal{M}_{0,0}$ is separated from the eigenvalue by a gap.

Since $\mathcal{M}_p = \oplus_a \mathcal{M}_{p,a}$, and using the uniform bound (C.28), we obtain immediately that the operator $\mathcal{M}_0$ has a simple eigenvalue at 0 with corresponding eigenvector ξ^{eq}

$$\xi^{eq} := \varphi^{eq} \oplus \underbrace{0 \oplus \ldots \oplus 0}_{a \neq 0}, \tag{C.29}$$

separated from the rest of the spectrum of $\mathcal{M}_0$ by a gap. By the analyticity in κ, see (4.6), and the correspondance between κ and (p, ν), as stated in (2.60), we can apply analytic perturbation theory in p to the family of operators $\mathcal{M}_p$. We conclude that for p in a neighborhood of 0, the operator $\mathcal{M}_p$ has a simple eigenvalue, which we call $f_{rw}(p)$, that is separated by a gap from the rest of the spectrum. We also obtain that the corresponding eigenvector is analytic in p and ν.

In this way we have derived all claims of Proposition 4.2, except for the symmetry $\nabla_p f_{rw}(p) = 0$ and the strict postive-definiteness of the matrix $(\nabla_p)^2 f_{rw}(p)$. These two claims will be proven in Section C.2.6. We note that the function $f_{rw}(p)$, which we defined above as the simple and isolated eigenvalue of $\mathcal{M}_p$ with maximal real part, is also a simple and isolated eigenvalue of $\mathcal{M}_{p,0}$ with maximal real part.

C.2.6 Strict positivity of the diffusion constant

By the remark at the end of Section C.2.5 and the fact that $\mathrm{sp}\hat{\mathcal{M}}_{p,0} = \mathrm{sp}\mathcal{M}_{p,0}$, we view $f_{rw}(p)$ as the eigenvalue of $\hat{\mathcal{M}}_{p,0}$ that reduces to 0 for $p = 0$.

We recall that $\hat{\mathcal{M}}_{p,0} = \hat{\mathcal{M}}_{0,0} + K_p$ and we define the operator-valued vector $V := \nabla_p K_p\big|_{p=0}$ (note that V is in fact a vector of operators). The first order shift of the eigenvalue is given by

$$\nabla_p f_{rw}(p) := \frac{1}{\langle \hat{\varphi}^{eq}, \hat{\varphi}^{eq} \rangle} \langle \hat{\varphi}^{eq}, V \hat{\varphi}^{eq} \rangle = 0 \tag{C.30}$$

To check that (C.30) indeed vanishes, we use that $\hat{\varphi}^{eq}$ is symmetric under the transformation $k \mapsto -k$ (in fact, it is independent of k) while V is anti-symmetric under $k \mapsto -k$ (this follows from the symmetry $\varepsilon(k) = \varepsilon(-k)$ in Assumption 2.1).

The second order shift is then given by

$$D_{rw} := (\nabla_p)^2 f_{rw}(p) = -\frac{1}{\langle \hat{\varphi}^{eq}, \hat{\varphi}^{eq} \rangle} \langle \hat{\varphi}^{eq}, V \hat{\mathcal{M}}_{0,0}^{-1} V \hat{\varphi}^{eq} \rangle + \frac{1}{\langle \hat{\varphi}^{eq}, \hat{\varphi}^{eq} \rangle} \langle \hat{\varphi}^{eq}, (\nabla_p)^2 K_p \hat{\varphi}^{eq} \rangle \tag{C.31}$$

where the first term on the RHS of (C.31) is well-defined since $V\hat{\varphi}^{eq}$ is orthogonal to the 0-spectral subspace of $\mathcal{M}_{0,0}$, by (C.30). The second term vanishes because $(\nabla_p)^2 K_p = 0$, as can again be checked explicitly.

Let $\upsilon \in \mathbb{R}^d$ and $V_\upsilon := \upsilon \cdot V$ (recall that V is a vector). Then, by (C.31),

$$\upsilon \cdot D_{rw} \upsilon = -\frac{1}{\langle \hat{\varphi}^{eq}, \hat{\varphi}^{eq} \rangle} \langle \hat{\varphi}^{eq}, V_\upsilon \hat{\mathcal{M}}_{0,0}^{-1} V_\upsilon \hat{\varphi}^{eq} \rangle \tag{C.32}$$

Upon using the spectral theorem and the gap for the self-adjoint operator $\hat{\mathcal{M}}_{0,0}$, we see that the RHS of the last expression is positive and it can only vanish if

$$0 = \| V_\upsilon \hat{\varphi}^{eq} \|^2 = \left[\sum_{e \in \mathrm{sp} Y} \mathrm{e}^{-\beta e} \right] \int \mathrm{d}k |\upsilon \cdot \nabla \varepsilon(k)|^2 \tag{C.33}$$

which is however excluded by Assumption 2.1. The strict positive-definiteness of the diffusion constant D_{rw} is hence proven.

D Appendix: Combinatorics

In this appendix, we show how to integrate over irreducible equivalence classes of diagrams. In other words, we assume that the $\underline{x}, \underline{l}$-coordinates have already been summed over (or a supremum over them has been taken) and we carry out the remaining integration over the time-coordinates $\underline{t}$ and the diagram size $|\sigma|$. We first define a function of diagrams, $K(\sigma)$, that depends only on the equivalence class $[\sigma]$. Let k be a positive function on $\mathbb{R}^+$ and put

$$K(\sigma) := \prod_{i=1}^{|\sigma|} k(v_i - u_i) \tag{D.1}$$

where (u_i, v_i) are the pairs of times in the diagram σ. In the applications, the function k will be (a multiple of) $\sup_x |\psi(x,t)|$, sometimes restricted to $t < \tau$ or $t > \tau$.

Lemma D.1. *Let $a \geq 0$ and assume that $\|t\mathrm{e}^{at}k\|_1 = \int_{\mathbb{R}^+} \mathrm{d}t\, t\mathrm{e}^{at}k(t) < 1$, then*

$$\int\limits_{\mathbb{R}^+} \mathrm{d}t\, \mathrm{e}^{at} \int\limits_{\Pi_T \Sigma_{[0,t]}(\mathrm{mir})} \mathrm{d}[\sigma] K(\sigma) \leq \|\mathrm{e}^{at}k\|_1 \frac{1}{1 - \|t\mathrm{e}^{at}k\|_1} \tag{D.2}$$

If in addition, $\|t\mathrm{e}^{\tilde{a}t}k\|_1 < 1$ with $\tilde{a} := a + \|k\|_1$, then

$$\int\limits_{\mathbb{R}^+} \mathrm{d}t\, \mathrm{e}^{at} \int\limits_{\Pi_T \Sigma_{[0,t]}(\mathrm{ir})} \mathrm{d}[\sigma]\, K(\sigma) \quad \leq \quad 2\|\mathrm{e}^{\tilde{a}t}k\|_1 \frac{1}{1 - \|t\mathrm{e}^{\tilde{a}t}k\|_1} \tag{D.3}$$

$$\int\limits_{\mathbb{R}^+} \mathrm{d}t\, \mathrm{e}^{at} \int\limits_{\Pi_T \Sigma_{[0,t]}(\mathrm{ir})} \mathrm{d}[\sigma] 1_{|\sigma| \geq 2}\, K(\sigma) \quad \leq \quad 2\|\mathrm{e}^{\tilde{a}t}k\|_1 \frac{\|t\mathrm{e}^{\tilde{a}t}k\|_1}{1 - \|t\mathrm{e}^{\tilde{a}t}k\|_1} \tag{D.4}$$

Proof. First, we note that for each irreducible diagram $\sigma \in \Sigma_{[0,t]}(\mathrm{ir})$, we can find a subdiagram $\sigma' \subset \sigma$ such that σ' is minimally irreducible in $[0,t]$, i.e., $\sigma' \in \Sigma_{[0,t]}(\mathrm{mir})$. Note that the choice of subdiagram σ' is not necessarily unique. Conversely, given a minimally irreducible diagram $\sigma' \in \Sigma_{[0,t]}(\mathrm{mir})$, we can add any diagram $\sigma'' \in \Sigma_{[0,t]}$ to σ', thereby creating a new irreducible diagram $\sigma := \sigma' \cup \sigma'' \in \Sigma_{[0,t]}(\mathrm{ir})$. By these considerations, we easily deduce

$$\int\limits_{\Pi_T \Sigma_{[0,t]}(\mathrm{ir})} \mathrm{d}[\sigma]\, 1_{|\sigma| \geq 2} K(\sigma) \quad \leq \quad \left(\int\limits_{\Pi_T \Sigma_{[0,t]}(\mathrm{mir})} \mathrm{d}[\sigma'] 1_{|\sigma'| \geq 2} K(\sigma') \right) \left(1 + \int\limits_{\Pi_T \Sigma_{[0,t]}} \mathrm{d}[\sigma''] K(\sigma'') \right) \tag{D.5}$$

$$+ \quad \left(\int\limits_{\Pi_T \Sigma_{[0,t]}(\mathrm{mir})} \mathrm{d}[\sigma'] 1_{|\sigma'| = 1} K(\sigma') \right) \left(\int\limits_{\Pi_T \Sigma_{[0,t]}} \mathrm{d}[\sigma''] K(\sigma'') \right) \tag{D.6}$$

The $1 + \cdot$ in (D.5) covers the case in which the diagram σ was itself minimally irreducible, and hence no diagrams σ'' are added to σ'. In (D.6), one always has to add at least one pair to σ', since $|\sigma| \geq 2$ but $|\sigma'| = 1$. In fact, the equivalence classes in the inequality could be dropped, i.e., one can omit the projections Π_T and replace $\mathrm{d}[\sigma], \mathrm{d}[\sigma'], \mathrm{d}[\sigma'']$ by $\mathrm{d}\sigma, \mathrm{d}\sigma', \mathrm{d}\sigma''$, respectively.

We recall that if a diagram σ with $|\sigma| = 1$ is irreducible (or minimally irreducible) in the interval I, then its time-coordinates are fixed to be the boundaries of I; i.e., there is only one equivalence class of such diagrams. Hence

$$\int\limits_{\Pi_T \Sigma_{[0,t]}(\mathrm{ir})} \mathrm{d}[\sigma] 1_{|\sigma|=1} K(\sigma) = \int\limits_{\Pi_T \Sigma_{[0,t]}(\mathrm{mir})} \mathrm{d}[\sigma] 1_{|\sigma|=1} K(\sigma) = k(t) \tag{D.7}$$

The unconstrained integral over all (equivalence classes of) diagrams, that appears in (D.5) and (D.6), can be

performed as follows

$$
\begin{aligned}
\int_{\Pi_T \Sigma_{[0,t]}} \mathrm{d}\sigma \mathrm{d}[\sigma] K(\sigma) &= \sum_{n\geq 1} \int_{0<u_1<\ldots<u_n<t} \mathrm{d}u_1 \ldots \mathrm{d}u_n \int_{v_i>u_i} \mathrm{d}v_1 \ldots \mathrm{d}v_n \left(\prod_{i=1}^n k(v_i-u_i) \right) \\
&\leq \sum_{n\geq 1} \int_{0<u_1<\ldots<u_n<t} \mathrm{d}u_1 \ldots \mathrm{d}u_n (\|k\|_1)^n = \sum_{n\geq 1} \frac{t^n}{n!} (\|k\|_1)^n = \mathrm{e}^{t\|k\|_1} - 1
\end{aligned} \tag{D.8}
$$

Next, we perform the integral over (equivalence classes of) minimally irreducible diagrams. For $\sigma \in \Sigma_{[0,t]}(\mathrm{mir})$ with $|\sigma| = n > 1$, the relative order of the times u_i, v_i is fixed as follows:

$$
0 = u_1 \leq u_2 \leq v_1 \leq u_3 \leq v_2 \leq u_4 \leq \ldots \leq v_{n-2} \leq u_n \leq v_{n-1} \leq v_n = t \tag{D.9}
$$

We have hence

$$
\begin{aligned}
\int_{\mathbb{R}^+} \mathrm{d}t\, \mathrm{e}^{at} \int_{\Pi_T \Sigma_{[0,t]}(\mathrm{mir})} \mathrm{d}\sigma K(\sigma) 1_{|\sigma|=n} &= \int_0^\infty \mathrm{d}v_1 k(v_1-u_1) \mathrm{e}^{a(v_1-u_1)} \int_0^{v_1} \mathrm{d}u_2 \int_{v_1}^\infty \mathrm{d}v_2 \ldots \\
\ldots \quad \int_{v_{n-5}}^{v_{n-4}} \mathrm{d}u_{n-2} \int_{v_{n-3}}^\infty \mathrm{d}v_{n-2} \ldots & \int_{v_{n-3}}^{v_{n-2}} \mathrm{d}u_{n-1} \int_{v_{n-2}}^\infty \mathrm{d}v_{n-1} \mathrm{e}^{a(v_{n-1}-v_{n-2})} k(v_{n-1}-u_{n-1}) \\
& \int_{v_{n-2}}^{v_{n-1}} \mathrm{d}u_n \int_{v_{n-1}}^\infty \mathrm{d}v_n \mathrm{e}^{a(v_n-v_{n-1})} k(v_n-u_n).
\end{aligned} \tag{D.10}
$$

First we extend the domain of integration of u_n from $[v_{n-2}, v_{n-1}]$ to $(-\infty, v_{n-1}]$ and we estimate the integrals over the variables u_n and v_n by

$$
\int_{-\infty}^{v_{n-1}} \mathrm{d}u_n \int_{v_{n-1}}^\infty \mathrm{d}v_n \mathrm{e}^{a(v_n-v_{n-1})} k(v_n-u_n) \leq \|t e^{at} k\|_1 \tag{D.11}
$$

Next, we perform the integration over u_{n-1}, v_{n-1} in the same way, we continue the procedure until only the variable v_1 is left (note that $u_1 = 0$ is fixed). The integral over v_1 gives $\|\mathrm{e}^{at} k\|_1$. This yields the bound

$$
\text{LHS of (D.10)} \; \leq \|\mathrm{e}^{at} k\|_1 \times \|t e^{at} k\|_1^{n-1} \tag{D.12}
$$

We are ready to evaluate the Laplace transform of (D.5)-(D.6). Using (D.8), we bound

$$
\left(1 + \int_{\Pi_T \Sigma_I} \mathrm{d}[\sigma] K(\sigma) \right) \leq \mathrm{e}^{t\|k\|_1}, \qquad \left(\int_{\Pi_T \Sigma_I} \mathrm{d}[\sigma] K(\sigma) \right) \leq \mathrm{e}^{t\|k\|_1} - 1 \leq t\|k\|_1 \mathrm{e}^{t\|k\|_1} \tag{D.13}
$$

Combining this with (D.7) and (D.12) , and summing over $n \geq 2$, we obtain

$$
\int \mathrm{d}t\, \mathrm{e}^{at} \int_{\Pi_T \Sigma_{[0,t]}(\mathrm{ir})} \mathrm{d}[\sigma] 1_{|\sigma|\geq 2}\, K(\sigma) \leq \|\mathrm{e}^{\tilde{a}t} k\|_1 \frac{\|t\mathrm{e}^{\tilde{a}t} k\|_1}{1 - \|t\mathrm{e}^{\tilde{a}t} k\|_1} + \|k\|_1 \|t\mathrm{e}^{\tilde{a}t} k\|_1 \tag{D.14}
$$

where the two terms on the RHS correspond to (D.5) and (D.6), respectively. This ends the proof of (D.4). The bound in (D.3) follows by adding $\|\mathrm{e}^{at} k\|_1$, which is the contribution of $|\sigma| = 1$ (see (D.7)), to (D.4). The bound (D.2) is proven by summing (D.12) over $n \geq 1$.

□

References

[1] R. Alicki and M. Fannes. *Quantum Dynamical Systems*. Oxford University Press, 2001.

[2] H. Araki and E. J. Woods. Representations of the canonical commutation relations describing a nonrelativistic infinite free Bose gas. *J. Math. Phys.*, 4:637, 1963.

[3] V. Bach, J. Fröhlich, and I. Sigal. Return to equilibrium. *J. Math. Phys.*, 41:3985, 2000.

[4] O. Brattelli and D. W. Robinson. *Operator Algebras and Quantum Statistical Mechanics: 2*. Springer-Verlag, Berlin, 2nd edition, 1996.

[5] W. Bryc. A remark on the connection between the large deviation principle and the central limit theorem. *Stat. and Prob. Lett.*, 18, 1993.

[6] L.A. Bunimovich and Ya. G. Sinai. Statistical properties of Lorentz gas with periodic configuration of scatterers. *Commun Math. Phys.*, 78:479–497, 1981.

[7] J. Clark, W. De Roeck, and C. Maes. Diffusive behaviour from a quantum master equation. *preprint arXiv:0812.2858*, 2008.

[8] E. B. Davies. *Linear Operators and their spectra*. Cambridge University Press, 2007.

[9] E.B. Davies. Markovian master equations. *Comm. Math. Phys.*, 39:91–110, 1974.

[10] J. Dereziński. *Introduction to Representations of Canonical Commutation and Anticommutation Relations*, volume 695 of *Lecture Notes in Physics*. Springer-Verlag, Berlin, 2006.

[11] J. Dereziński and R. Früboes. Fermi golden rule and open quantum systems. In S. Attal, A. Joye, and C.-A. Pillet, editors, *Lecture notes Grenoble Summer School on Open Quantum Systems*, Lecture Notes in Mathematics. Springer, 2003.

[12] J. Dereziński and V. Jakšić. Return to equilibrium for Pauli-Fierz systems. *Ann. H. Poincaré*, 4:739–793, 2003.

[13] M. Disertori, T. Spencer, and M. Zirnbauer. Quasi-diffusion in a 3d supersymmetric hyperbolic sigma model. *arXiv:0901.1652*.

[14] L. Erdös. Linear Boltzmann equation as the long time dynamics of an electron weakly coupled to a phonon field. *J. Stat. Phys.*, 107(85):1043–1127, 2002.

[15] L. Erdös, M. Salmhofer, and H.-T. Yau. Quantum diffusion of the random Schrödinger evolution in the scaling limit ii. the recollision diagrams. *Comm. Math. Phys*, 271:1–53, 2007.

[16] L. Erdös, M. Salmhofer, and H.-T. Yau. Quantum diffusion of the random Schrödinger evolution in the scaling limit i. the non-recollision diagrams. *Acta Mathematica*, 200:211–277, 2008.

[17] L. Erdös and H.-T. Yau. Linear Boltzmann equation as the weak coupling limit of a random Schrödinger equation. *Comm. Pure Appl. Math.*, 53(6):667 – 735, 2000.

[18] L.C. Evans. *Partial Differential Equations (Graduate Studies in Mathematics, V. 19) GSM/19*. American Mathematical Society, 1998.

[19] A. Frigerio. Stationary states of quantum dynamical semigroups. *Comm. Math. Phys.*, 63 (3):269–276, 1978.

[20] J. Fröhlich and M. Merkli. Another return of 'return to equilibrium'. *Comm. Math. Phys.*, 251:235–262, 2004.

[21] A. S. Holevo. A note on covariant dynamical semigroups. *Rep. Math. Phys.*, 32:211–216, 1992.

[22] L. Van Hove. Quantum-mechanical perturbations giving rise to a statistical transport equation. *Physica*, 21:517–540, 1955.

[23] V. Jakšić and C.-A. Pillet. On a model for quantum friction. iii: Ergodic properties of the spin-boson system. *Comm. Math. Phys.*, 178:627–651, 1996.

[24] J.Bricmont and A. Kupiainen. Coupled analytic maps. *Nonlinearity*, 8:379–393, 1995.

[25] Y. Kang and J. Schenker. Diffusion of wave packets in a Markov random potential. *arXiv:0808.2784*, 2008.

[26] T. Kato. *Perturbation Theory for Linear Operators*. Springer, Berlin, second edition, 1976.

[27] A. Knauf. Ergodic and topological properties of coulombic periodic potentials. *Comm. Math. Phys.*, 110(1):89–112.

[28] T. Komorowski and L. Ryzhik. Diffusion in a weakly random hamiltonian flow. *Comm. Math. Phys*, 263:277–323, 2006.

[29] J. Lebowitz and H. Spohn. Irreversible thermodynamics for quantum systems weakly coupled to thermal reservoirs. *Adv. Chem. Phys.*, 39:109–142, 1978.

[30] A. A. Ovchinnikov and N. S. Erikhman. Motion of a quantum particle in a stochastic medium. *Sov. Phys.-JETP*, 40:733–737, 1975.

[31] W. De Roeck. Large deviation generating function for currents in the Pauli-Fierz model. *Rev. Math. Phys*, 21(4):549–585, 2009.

[32] W. De Roeck, J. Fröhlich, and A. Pizzo. Quantum Brownian motion in a simple model system. *Comm. Math. Phys*, 293(2):361–398, 2010.

[33] W. De Roeck and D. Spehner. Derivation of the quantum master equation for massive tracer particles. *in preparation*.

[34] H. Spohn. An algebraic condition for the approach to equilibrium of an open N-level system. *Lett. Math. Phys.*, 2:33–38, 1977.

[35] B. Vacchini and K. Hornberger. Quantum linear boltzmann equation. *arXiv:0904.3911*, 2009.